Contents 29

Integral Vector Calculus

Learning outcomes

In this Workbook you will learn how to integrate functions involving vectors. You will learn how to evaluate line integrals i.e. where a scalar or a vector is summed along a line or contour. You will be able to evaluate surface and volume integrals where a function involving vectors is summed over a surface or volume. You will learn about some theorems relating to line, surface or volume integrals viz Stokes' theorem, Gauss' divergence theorem and Green's theorem.

Line Integrals

29.1

Introduction

HELM workbook 28 considered the differentiation of scalar and vector fields. Here we consider how to integrate such fields along a line. Firstly, integrals involving scalars along a line will be considered. Subsequently, line integrals involving vectors will be considered. These can give scalar or vector answers depending on the form of integral involved. Of particular interest are the integrals of conservative vector fields.

Prerequisites

Before starting this Section you should ...

- have a thorough understanding of the basic techniques of integration
- be familiar with the operators div, grad and curl

Learning Outcomes

On completion you should be able to ...

- integrate a scalar or vector quantity along a line

1. Line integrals

HELM 28 was concerned with evaluating an integral over **all** points within a rectangle or other shape (or over a cuboid or other volume). In a related manner, an integral can take place over a line or curve running through a two-dimensional (or three-dimensional) region. Line integrals may involve scalar or vector fields. Those involving scalar fields are dealt with first.

Line integrals in two dimensions

A line integral in two dimensions may be written as

$$\int_C F(x,y)dw$$

There are three main features determining this integral:

$F(x,y)$: This is the scalar function to be integrated e.g. $F(x,y) = x^2 + 4y^2$.

C: This is the curve along which integration takes place. e.g. $y = x^2$ or $x = \sin y$ or $x = t - 1;\ y = t^2$ (where x and y are expressed in terms of a parameter t).

dw: This gives the variable of the integration. Three main cases are dx, dy and ds. Here 's' is arc length and so indicates position along the curve C.

ds may be written as $ds = \sqrt{(dx)^2 + (dy)^2}$ or $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$.

A fourth case is when $F(x,y)\ dw$ has the form: $F_1dx + F_2dy$. This is a combination of the cases dx and dy.

The integral $\displaystyle\int_C F(x,y)\ ds$ represents the area beneath the surface $z = F(x,y)$ but above the curve C.

The integrals $\displaystyle\int_C F(x,y)\ dx$ and $\displaystyle\int_C F(x,y)\ dy$ represent the projections of this area onto the xz and yz planes respectively.

A particular case of the integral $\displaystyle\int_C F(x,y)\ ds$ is the integral $\displaystyle\int_C 1\ ds$. This is a means of calculating the length along a curve i.e. an arc length.

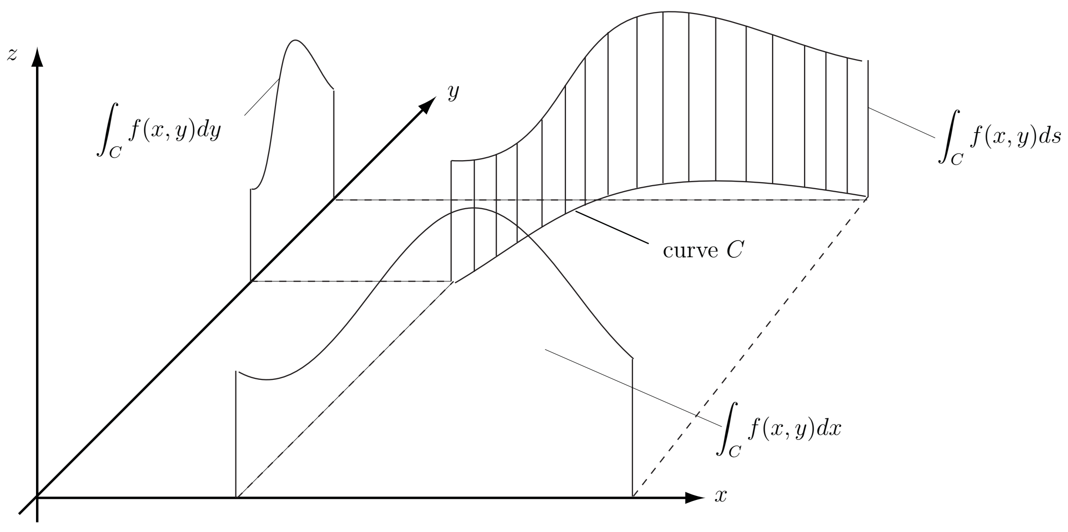

Figure 1: Representation of a line integral and its projections onto the xz and yz planes

The technique for evaluating a line integral is to express all quantities in the integral in terms of a single variable. If the integral is with respect to 'x' or 'y', then the curve 'C' and the function 'F' may be expressed in terms of the relevant variable. If the integral is with respect to ds, normally all quantities are expressed in terms of x. If x and y are given in terms of a parameter t, then t is used as the variable.

Example 1

Find $\displaystyle\int_c x\,(1+4y)\,dx$ where C is the curve $y=x^2$, starting from $x=0, y=0$ and ending at $x=1, y=1$.

Solution

As this integral concerns only points along C and the integration is carried out with respect to x, y may be replaced by x^2. The limits on x will be 0 to 1. So the integral becomes

$$\begin{aligned}\int_C x(1+4y)\,dx &= \int_{x=0}^{1} x\left(1+4x^2\right)dx = \int_{x=0}^{1}\left(x+4x^3\right)dx \\ &= \left[\frac{x^2}{2}+x^4\right]_0^1 = \left(\frac{1}{2}+1\right)-(0) = \frac{3}{2}\end{aligned}$$

Example 2

Find $\displaystyle\int_c x\,(1+4y)\;dy$ where C is the curve $y=x^2$, starting from $x=0, y=0$ and ending at $x=1, y=1$. This is the same as Example 1 other than dx being replaced by dy.

Solution

As this integral concerns only points along C and the integration is carried out with respect to y, everything may be expressed in terms of y, i.e. x may be replaced by $y^{1/2}$. The limits on y will be 0 to 1. So the integral becomes

$$\begin{aligned}\int_C x(1+4y)\,dy &= \int_{y=0}^{1} y^{1/2}\,(1+4y)\,dx = \int_{y=0}^{1}\left(y^{1/2}+4y^{3/2}\right)dx \\ &= \left[\frac{2}{3}y^{3/2}+\frac{8}{5}x^{5/2}\right]_0^1 = \left(\frac{2}{3}+\frac{8}{5}\right)-(0) = \frac{34}{15}\end{aligned}$$

Example 3

Find $\int_c x\,(1+4y)\,ds$ where C is the curve $y = x^2$, starting from $x = 0, y = 0$ and ending at $x = 1, y = 1$. This is the same integral and curve as the previous two examples but the integration is now carried out with respect to s, the arc length parameter.

Solution

As this integral is with respect to x, all parts of the integral can be expressed in terms of x, Along $y = x^2$, $\quad ds = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}dx = \sqrt{1+(2x)^2}dx = \sqrt{1+4x^2}dx$

So, the integral is

$$\int_c x\,(1+4y)\,ds = \int_{x=0}^{1} x\left(1+4x^2\right)\sqrt{1+4x^2}\;dx = \int_{x=0}^{1} x\left(1+4x^2\right)^{3/2}dx$$

This can be evaluated using the transformation $u = 1+4x^2$ so $du = 8xdx$ i.e. $x\;dx = \dfrac{du}{8}$. When $x = 0,\;\; u = 1$ and when $x = 1,\;\; u = 5$.

Hence,

$$\begin{aligned}\int_{x=0}^{1} x\left(1+4x^2\right)^{3/2}dx &= \frac{1}{8}\int_{u=1}^{5} u^{3/2}du \\ &= \frac{1}{8}\times\frac{2}{5}\left[\;u^{5/2}\right]_1^5 = \frac{1}{20}\left[5^{5/2}-1\right]\approx 2.745\end{aligned}$$

Note that the results for Examples 1,2 and 3 are all different: Example 3 is the area between a curve and a surface above; Examples 1 and 2 give projections of this area onto other planes.

Example 4

Find $\int_C xy\;dx$ where, on C, x and y are given in terms of a parameter t by

$x = 3t^2$, $y = t^3 - 1$ for t varying from 0 to 1.

Solution

Everything can be expressed in terms of t, the parameter. Here $x = 3t^2$ so $dx = 6t\;dt$. The limits on t are $t = 0$ and $t = 1$. The integral becomes

$$\begin{aligned}\int_C xy\;dx &= \int_{t=0}^{1} 3t^2\,(t^3-1)\,6t\;dt = \int_{t=0}^{1}(18t^6-18t^3)\;dt \\ &= \left[\frac{18}{7}t^7 - \frac{18}{4}t^4\right]_0^1 = \frac{18}{7}-\frac{9}{2}-0 = -\frac{27}{14}\end{aligned}$$

Key Point 1

A line integral is normally evaluated by expressing all variables in terms of one variable.

In general

$$\int_C f(x,y)\ ds \neq \int_C f(x,y)\ dy \neq \int_C f(x,y)\ dx$$

Task

For $F(x,y) = 2x + y^2$, find (i) $\int_C F(x,y)\ dx$, (ii) $\int_C F(x,y)\ dy$, (iii) $\int_C F(x,y)\ ds$ where C is the line $y = 2x$ from $(0,0)$ to $(1,2)$.

Express each integral as a simple integral with respect to a single variable and hence evaluate each integral:

Your solution

Answer

(i) $\displaystyle\int_{x=0}^{1} (2x + 4x^2)\ dx = \frac{7}{3}$, (ii) $\displaystyle\int_{y=0}^{2} (y + y^2)\ dy = \frac{14}{3}$, (iii) $\displaystyle\int_{x=0}^{1} (2x + 4x^2)\sqrt{5}\ dx = \frac{7}{3}\sqrt{5}$

Task

Find (i) $\int_C F(x,y)\,dx$, (ii) $\int_C F(x,y)\,dy$, (iii) $\int_C F(x,y)\,ds$ where $F(x,y) = 1$ and C is the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ from $(1, \frac{1}{2})$ to $(2, 2 - \frac{1}{4}\ln 2)$.

Your solution

Answer

(i) $\displaystyle\int_1^2 1\,dx = 1$, (ii) $\displaystyle\int_{1/2}^{2-(1/4)\ln 2} 1\,dy = \frac{3}{2} - \frac{1}{4}\ln 2$, (iii) $y = \dfrac{1}{2}x^2 - \dfrac{1}{4}\ln x \Rightarrow \dfrac{dy}{dx} = x - \dfrac{1}{4x}$

$$\therefore \int 1\,ds = \int_1^2 \sqrt{1 + (x - \frac{1}{4x})^2}\,dx = \int_1^2 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}\,dx = \int_1^2 (x + \frac{1}{4x})\,dx = \frac{3}{2} + \frac{1}{4}\ln 2.$$

Task

Find (i) $\int_C F(x,y)\,dx$, (ii) $\int_C F(x,y)\,dy$, (iii) $\int_C F(x,y)\,ds$

where $F(x,y) = \sin 2x$ and C is the curve $y = \sin x$ from $(0,0)$ to $(\dfrac{\pi}{2}, 1)$.

Your solution

Answer

(i) $\displaystyle\int_0^{\pi/2} \sin 2x\,dx = 1$, (ii) $\displaystyle\int_0^{\pi/2} 2\sin x\cos^2 x\,dx = \frac{2}{3}$

(iii) $\displaystyle\int_0^{\pi/2} \sin 2x\sqrt{1+\cos^2 x}\,dx = \frac{2}{3}(2\sqrt{2} - 1)$, using the substitution $u = 1 + \cos^2 x$.

2. Line integrals of scalar products

Integrals of the form $\int_C \underline{F} \cdot \underline{dr}$ occur in applications such as the following.

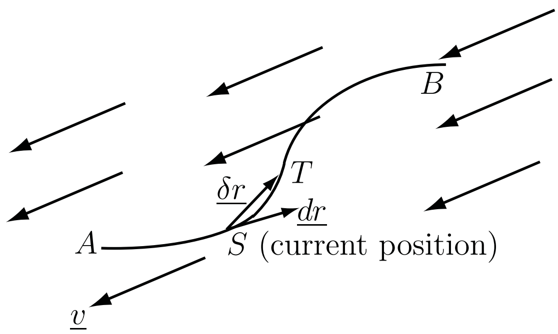

Figure 2: Schematic for cyclist travelling from A to B into a head wind

Consider a cyclist riding along the road from A to B (Figure 2). Suppose it is necessary to find the total work the cyclist has to do in overcoming a wind of velocity $\underline{v}$.

On moving from S to T, along an element $\underline{\delta r}$ of road, the work done is given by 'Force $\times$ distance' $= |\underline{F}| \times |\underline{\delta r}| \cos\theta$ where $\underline{F}$, the force, is directly proportional to $\underline{v}$, but in the opposite direction, and $|\underline{\delta r}| \cos\theta$ is the component of the distance travelled in the direction of the wind.
So, the work done travelling $\underline{\delta r}$ is $-k\underline{v} \cdot \underline{\delta r}$. Letting $\underline{\delta r}$ become infinitesimally small, the work done becomes $-k\underline{v} \cdot \underline{dr}$ and the total work is $-k\int_A^B \underline{v} \cdot \underline{dr}$.
This is an example of the integral along a line, of the scalar product of a vector field, with a vector element of the line. The term **scalar line integral** is often used for integrals of this form. The vector $\underline{dr}$ may be considered to be $dx\ \underline{i} + dy\ \underline{j} + dz\ \underline{k}$.

Multiplying out the scalar product, the 'scalar line integral' of the vector $\underline{F}$ along contour C, is given by $\int_C \underline{F} \cdot \underline{dr}$ and equals $\int_C \{F_x\ dx + F_y\ dy + F_z\ dz\}$ in three dimensions, and $\int_C \{F_x\ dx + F_y\ dy\}$ in two dimensions, where F_x, F_y, F_z are the components of $\underline{F}$.

If the contour C has its start and end points in the same positions i.e. it represents a closed contour, the symbol $\oint_C$ rather than $\int_C$ is used, i.e. $\oint_C \underline{F} \cdot \underline{dr}$.
As before, to evaluate the line integral, express the path and the function $\underline{F}$ in terms of either x, y and z, or in terms of a parameter t. Note that t often represents time.

Example 5

Find $\int_C \{2xy\ dx - 5x\ dy\}$ where C is the curve $y = x^3 \quad 0 \leq x \leq 1$.

[This is the integral $\int_C \underline{F} \cdot \underline{dr}$ where $\underline{F} = 2xy\underline{i} - 5x\underline{j}$ and $\underline{dr} = dx\ \underline{i} + dy\ \underline{j}$.]

Solution

It is possible to split this integral into two different integrals and express the first term as a function of x and the second term as a function of y. However, it is also possible to express everything in terms of x. Note that on C, $y = x^3$ so $dy = 3x^2\,dx$ and the integral becomes

$$\begin{aligned}\int_C \{2xy\,dx - 5x\,dy\} &= \int_{x=0}^{1} \left(2x\,x^3\,dx - 5x\,3x^2\,dx\right) = \int_0^1 (2x^4 - 15x^3)\,dx \\ &= \left[\frac{2}{5}x^5 - \frac{15}{4}x^4\right]_0^1 = \frac{2}{5} - \frac{15}{4} - 0 = -\frac{67}{20}\end{aligned}$$

Key Point 2

An integral of the form $\displaystyle\int_C \underline{F}\cdot\underline{dr}$ may be expressed as $\displaystyle\int_C \{F_x\,dx + F_y\,dy + F_z\,dz\}$. Knowing the expression for the path C, every term in the integral can be further expressed in terms of one of the variables x, y or z or in terms of a parameter t and hence integrated.

If an integral is two-dimensional there are no terms involving z.

The integral $\displaystyle\int_C \underline{F}\cdot\underline{dr}$ evaluates to a scalar.

Example 6

Three paths from $(0,0)$ to $(1,2)$ are defined by

(a) $C_1: \quad y = 2x$
(b) $C_2: \quad y = 2x^2$
(c) $C_3: \quad y = 0$ from $(0,0)$ to $(1,0)$ and $x = 1$ from $(1,0)$ to $(1,2)$

Sketch each path, and along each path find $\displaystyle\int \underline{F}\cdot\underline{dr}$, where $\underline{F} = y^2\underline{i} + xy\underline{j}$.

Solution

(a) $\int \underline{F} \cdot \underline{dr} = \int \{y^2 dx + xydy\}$. Along $y = 2x$, $\dfrac{dy}{dx} = 2$ so $dy = 2dx$. Then

$$\begin{aligned}\int_{C_1} \underline{F} \cdot \underline{dr} &= \int_{x=0}^{1} \left\{(2x)^2\, dx + x\,(2x)\,(2dx)\right\} \\ &= \int_0^1 \left(4x^2 + 4x^2\right) dx = \int_0^1 8x^2 dx = \left[\frac{8}{3}x^2\right]_0^1 = \frac{8}{3}\end{aligned}$$

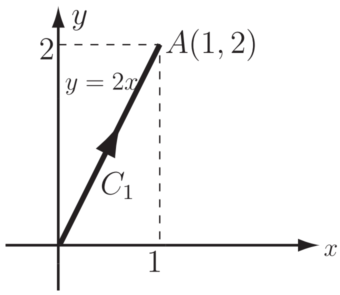

Figure 3(a): Integration along path C_1

(b) $\int \underline{F} \cdot \underline{dr} = \int \{y^2 dx + xydy\}$. Along $y = 2x^2$, $\dfrac{dy}{dx} = 4x$ so $dy = 4xdx$. Then

$$\int_{C_2} \underline{F} \cdot \underline{dr} = \int_{x=0}^{1} \left\{\left(2x^2\right)^2 dx + x\left(2x^2\right)(4xdx)\right\} = \int_0^1 12x^4 dx = \left[\frac{12}{5}x^5\right]_0^1 = \frac{12}{5}$$

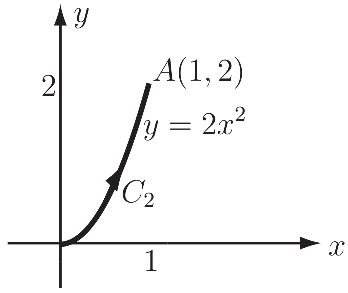

Figure 3(b): Integration along path C_2

Note that the answer is different to part (a), i.e., the line integral depends upon the path taken.

(c) As the contour C_3, has two distinct parts with different equations, it is necessary to break the full contour OA into the two parts, namely OB and BA where B is the point $(1, 0)$. Hence

$$\int_{C_3} \underline{F} \cdot \underline{dr} = \int_O^B \underline{F} \cdot \underline{dr} + \int_B^A \underline{F} \cdot \underline{dr}$$

Solution (contd.)

Along OB, $y = 0$ so $dy = 0$. Then

$$\int_O^B \underline{F} \cdot \underline{dr} = \int_{x=0}^1 \left(0^2 dx + x \times 0 \times 0\right) = \int_0^1 0dx = 0$$

Along AB, $x = 1$ so $dx = 0$. Then

$$\int_A^B \underline{F} \cdot \underline{dr} = \int_{y=0}^2 \left(y^2 \times 0 + 1 \times y \times dy\right) = \int_0^2 ydy = \left[\frac{1}{2}y^2\right]_0^2 = 2.$$

Hence $\displaystyle\int_{C_3} \underline{F} \cdot \underline{dr} = 0 + 2 = 2$

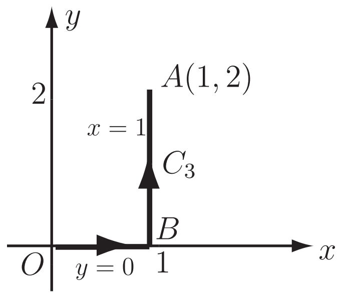

Figure 3(c): Integration along path C_3

Once again, the result is path dependent.

Key Point 3

In general, the value of a line integral depends on the path of integration as well as upon the end points.

Example 7

Find $\displaystyle\int_A^O \underline{F}\cdot\underline{dr}$, where $\underline{F} = y^2\underline{i} + xy\underline{j}$ (as in Example 6) and the path C_4 from A to O is the straight line from $(1,2)$ to $(0,0)$, that is the reverse of C_1 in Example 6(a).

Deduce $\displaystyle\oint_C \underline{F}\cdot\underline{dr}$, the integral around the closed path C formed by the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$ and the line $y = 2x$ from $(1,2)$ to $(0,0)$.

Solution

Reversing the path interchanges the limits of integration, which results in a change of sign for the value of the integral.

$$\int_A^O \underline{F}\cdot\underline{dr} = -\int_O^A \underline{F}\cdot\underline{dr} = -\frac{8}{3}$$

The integral along the parabola (calculated in Example 6(b)) evaluates to $\dfrac{12}{5}$, then

$$\oint_C \underline{F}\cdot\underline{dr} = \int_{C_2} \underline{F}\cdot\underline{dr} + \int_{C_4} \underline{F}\cdot\underline{dr} = \frac{12}{5} - \frac{8}{3} = -\frac{4}{15} \approx -0.267$$

Example 8

Consider the vector field

$$\underline{F} = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$$

Let C_1 and C_2 be the curves from $O = (0,0,0)$ to $A = (1,1,1)$, given by

$$\begin{array}{llll} C_1 : & x = t, \quad y = t, \quad z = t & (0 \leq t \leq 1) \\ C_2 : & x = t^2, \quad y = t, \quad z = t^2 & (0 \leq t \leq 1) \end{array}$$

(a) Evaluate the scalar integral of the vector field along each path.

(b) Find the value of $\displaystyle\oint_C \underline{F}\cdot\underline{dr}$ where C is the closed path along C_1 from O to A and back along C_2 from A to O.

Solution

(a) The path C_1 is given in terms of the parameter t by $x = t$, $y = t$ and $z = t$. Hence

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1 \text{ and } \frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k} = \underline{i} + \underline{j} + \underline{k}$$

Now by substituting for $x = y = z = t$ in $\underline{F}$ we have

$$\underline{F} = t^5\underline{i} + 2t^5\underline{j} + 3t^5\underline{k}$$

Hence $\underline{F} \cdot \dfrac{d\underline{r}}{dt} = t^5 + 2t^5 + 3t^5 = 6t^5$. The values of $t = 0$ and $t = 1$ correspond to the start and end point of C_1 and so these are the required limits of integration. Now

$$\int_{C_1} \underline{F} \cdot d\underline{r} = \int_0^1 \underline{F} \cdot \frac{d\underline{r}}{dt} dt = \int_0^1 6t^5 dt = \Big[t^6 \Big]_0^1 = 1$$

For the path C_2 the parameterisation is $x = t^2$, $y = t$ and $z = t^2$ so $\dfrac{d\underline{r}}{dt} = 2t\underline{i} + \underline{j} + 2t\underline{k}$. Substituting $x = t^2$, $y = t$ and $z = t^2$ in $\underline{F}$ we have

$$\underline{F} = t^8\underline{i} + 2t^9\underline{j} + 3t^8\underline{k} \text{ and } \underline{F} \cdot \frac{d\underline{r}}{dt} = 2t^9 + 2t^9 + 6t^9 = 10t^9$$

$$\int_{C_2} \underline{F} \cdot d\underline{r} = \int_0^1 10t^9 dt = \Big[t^{10} \Big]_0^1 = 1$$

(b) For the closed path C

$$\oint_C \underline{F} \cdot d\underline{r} = \int_{C_1} \underline{F} \cdot d\underline{r} - \int_{C_2} \underline{F} \cdot d\underline{r} = 1 - 1 = 0$$

(Note: A line integral round a closed path is not necessarily zero - see Example 7.)

Further points on Example 8

Vector Field	Path	Line Integral
$\underline{F}$	C_1	1
$\underline{F}$	C_2	1
$\underline{F}$	closed	0

Note that the value of the line integral of $\underline{F}$ is 1 for both paths C_1 and C_2. In fact, this result would hold for any path from $(0, 0, 0)$ to $(1, 1, 1)$.

The field $\underline{F}$ is an example of a **conservative vector field**; these are discussed in detail in the next subsection.

In $\displaystyle\int_C \underline{F} \cdot d\underline{r}$, the vector field $\underline{F}$ may be the gradient of a scalar field or the curl of a vector field.

Task

Consider the vector field

$$\underline{G} = x\underline{i} + (4x - y)\underline{j}$$

Let C_1 and C_2 be the curves from $O = (0, 0, 0)$ to $A = (1, 1, 1)$, given by

$$\begin{array}{llllll} C_1 & : & x = t, & y = t, & z = t & (0 \leq t \leq 1) \\ C_2 & : & x = t^2, & y = t, & z = t^2 & (0 \leq t \leq 1) \end{array}$$

(a) Evaluate the scalar integral $\displaystyle\int_C \underline{G} \cdot \underline{dr}$ of each vector field along each path.

(b) Find the value of $\displaystyle\oint_C \underline{G} \cdot \underline{dr}$ where C is the closed path along C_1 from O to A and back along C_2 from A to O.

Your solution

Answer

(a) The path C_1 is given in terms of the parameter t by $x = t$, $y = t$ and $z = t$. Hence

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1 \text{ and } \frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k} = \underline{i} + \underline{j} + \underline{k}$$

Substituting for $x = y = z = t$ in $\underline{G}$ we have

$$\underline{G} = t\underline{i} + 3t\underline{j} \text{ and } \underline{G} \cdot \frac{d\underline{r}}{dt} = t + 3t = 4t$$

The limits of integration are $t = 0$ and $t = 1$, then

$$\int_{C_1} \underline{G} \cdot d\underline{r} = \int_0^1 \underline{G} \cdot \frac{d\underline{r}}{dt} dt = \int_0^1 4t dt = \Big[2t^2\Big]_0^1 = 2$$

For the path C_2 the parameterisation is $x = t^2$, $y = t$ and $z = t^2$ so $\dfrac{d\underline{r}}{dt} = 2t\underline{i} + \underline{j} + 2t\underline{k}$.

Substituting $x = t^2$, $y = t$ and $z = t^2$ in $\underline{G}$ we have

$$\underline{G} = t^2\underline{i} + \left(4t^2 - t\right)\underline{j} \text{ and } \underline{G} \cdot \frac{d\underline{r}}{dt} = 2t^3 + 4t^2 - t$$

$$\int_{C_2} \underline{G} \cdot d\underline{r} = \int_0^1 \left(2t^3 + 4t^2 - t\right) dt = \left[\frac{1}{2}t^4 + \frac{4}{3}t^3 - \frac{1}{2}t^2\right]_0^1 = \frac{4}{3}$$

(b) For the closed path C $\quad \displaystyle\oint_C \underline{G} \cdot d\underline{r} = \int_{C_1} \underline{G} \cdot d\underline{r} - \int_{C_2} \underline{G} \cdot d\underline{r} = 2 - \frac{4}{3} = \frac{2}{3}$

(Note: The value of the integral around the closed path is non-zero, unlike Example 8.)

Example 9

Find $\displaystyle\int_C \left\{\underline{\nabla}(x^2y)\right\} \cdot d\underline{r}$ where C is the contour $y = 2x - x^2$ from $(0, 0)$ to $(2, 0)$.

Here, $\underline{\nabla}$ refers to the gradient operator, i.e. $\underline{\nabla}\phi \equiv \text{grad } \phi$

Solution

Note that $\underline{\nabla}(x^2y) = 2xy\underline{i} + x^2\underline{j}$ so the integral is $\displaystyle\int_C \left\{2xy\,dx + x^2\,dy\right\}$.

On $y = 2x - x^2$, $dy = (2 - 2x)\,dx$ so the integral becomes

$$\begin{aligned} \int_C \left\{2xy\,dx + x^2\,dy\right\} &= \int_{x=0}^2 \left\{2x(2x - x^2)\,dx + x^2(2 - 2x)\,dx\right\} \\ &= \int_0^2 (6x^2 - 4x^3)\,dx = \Big[2x^3 - x^4\Big]_0^2 = 0 \end{aligned}$$

Example 10

Two paths from $(0,0)$ to $(4,2)$ are defined by

(a) $C_1: \quad y = \dfrac{1}{2}x \qquad 0 \leq x \leq 4$

(b) C_2: The straight line $y = 0$ from $(0,0)$ to $(4,0)$ followed by
C_3: The straight line $x = 4$ from $(4,0)$ to $(4,2)$

For each path find $\displaystyle\int_C \underline{F} \cdot \underline{dr}$, where $\underline{F} = 2x\underline{i} + 2y\underline{j}$.

Solution

(a) For the straight line $y = \dfrac{1}{2}x$ we have $dy = \dfrac{1}{2}dx$

Then, $\displaystyle\int_{C_1} \underline{F} \cdot \underline{dr} = \int_{C_1} 2x\,dx + 2y\,dy = \int_0^4 \left(2x + \frac{x}{2}\right)\,dx = \int_0^4 \frac{5x}{2}\,dx = 20$

(b) For the straight line from $(0,0)$ to $(4,0)$ we have $\displaystyle\int_{C_2} \underline{F} \cdot \underline{dr} = \int_0^4 2x\,dx = 16$

For the straight line from $(4,0)$ to $(4,2)$ we have $\displaystyle\int_{C_3} \underline{F} \cdot \underline{dr} = \int_0^2 2y\,dy = 4$

Adding these two results gives $\displaystyle\int_C \underline{F} \cdot \underline{dr} = 16 + 4 = 20$

Task

Evaluate $\displaystyle\int_C \underline{F} \cdot \underline{dr}$, where $\underline{F} = (x-y)\underline{i} + (x+y)\underline{j}$ along each of the following paths

(a) C_1 : from $(1,1)$ to $(2,4)$ along the straight line $y = 3x - 2$:

(b) C_2 : from $(1,1)$ to $(2,4)$ along the parabola $y = x^2$:

(c) C_3 : along the straight line $x = 1$ from $(1,1)$ to $(1,4)$ then along the straight line $y = 4$ from $(1,4)$ to $(2,4)$.

Your solution

Answer

(a) $\int_1^2 (10x - 4)\, dx = 11,$

(b) $\int_1^2 (x + x^2 + 2x^3)\, dx = \frac{34}{3}$, (this differs from (a) showing path dependence)

(c) $\int_1^4 (1 + y)\, dy + \int_1^2 (x - 4)\, dx = 8$

Task

For the function $\underline{F}$ and paths in the last Task, deduce $\oint \underline{F} \cdot \underline{dr}$ for the closed paths

(a) C_1 followed by the reverse of C_2.

(b) C_2 followed by the reverse of C_3.

(c) C_3 followed by the reverse of C_1.

Your solution

Answer

(a) $-\frac{1}{3}$, (b) $\frac{10}{3}$, (c) -3. (note that all these are non-zero.)

Exercises

1. Consider $\int_C \underline{F}\cdot\underline{dr}$, where $\underline{F} = 3x^2y^2\underline{i} + (2x^3y-1)\underline{j}$. Find the value of the line integral along each of the paths from $(0,0)$ to $(1,4)$.

   (a) $y = 4x$ (b) $y = 4x^2$ (c) $y = 4x^{1/2}$ (d) $y = 4x^3$

2. Consider the vector field $\underline{F} = 2x\underline{i} + (xz-2)\underline{j} + xy\underline{k}$ and the two curves between $(0,0,0)$ and $(1,-1,2)$ defined by

   $C_1 : x = t^2,\ y = -t,\ z = 2t$ for $0 \leq t \leq 1$.
   $C_2 : x = t-1,\ y = 1-t,\ z = 2t-2$ for $1 \leq t \leq 2$.

   (a) Find $\int_{C_1} \underline{F}\cdot\underline{dr}$, $\int_{C_2} \underline{F}\cdot\underline{dr}$

   (b) Find $\oint_C \underline{F}\cdot\underline{dr}$ where C is the closed path from $(0,0,0)$ to $(1,-1,2)$ along C_1 and back to $(0,0,0)$ along C_2.

3. Consider the vector field $\underline{G} = x^2z\underline{i} + y^2z\underline{j} + \frac{1}{3}(x^3+y^3)\underline{k}$ and the two curves between $(0,0,0)$ and $(1,-1,2)$ defined by

   $C_1 : x = t^2,\ y = -t,\ z = 2t$ for $0 \leq t \leq 1$.
   $C_2 : x = t-1,\ y = 1-t,\ z = 2t-2$ for $1 \leq t \leq 2$.

   (a) Find $\int_{C_1} \underline{G}\cdot\underline{dr}$, $\int_{C_2} \underline{G}\cdot\underline{dr}$

   (b) Find $\oint_C \underline{G}\cdot\underline{dr}$ where C is the closed path from $(0,0,0)$ to $(1,-1,2)$ along C_1 and back to $(0,0,0)$ along C_2.

4. Find $\int_C \underline{F}\cdot\underline{dr})$ along $y = 2x$ from $(0,0)$ to $(2,4)$ for

   (a) $\underline{F} = \underline{\nabla}(x^2y)$

   (b) $\underline{F} = \underline{\nabla}\times(\frac{1}{2}x^2y^2\underline{k})$ [Here $\underline{\nabla}\times\underline{f}$ represents the curl of $\underline{f}$]

Answers

1. All are 12, and in fact the integral would be 12 for **any** path from (0,0) to (1,4).

2 (a) $2,\ \frac{5}{3}$ (b) $\frac{1}{3}$.

3 (a) $0,\ 0$ (b) 0.

4. (a) $\int_C 2xy\,dx + x^2\,dy = 16$, (b) $\int_C x^2y\,dx - xy^2\,dy = -24$.

3. Conservative vector fields

For some line integrals in the previous section, the value of the integral depended only on the vector field $\underline{F}$ and the start and end points of the line but not on the actual path between the start and end points. However, for other line integrals, the result depended on the actual details of the path of the line.

Vector fields are classified according to whether the line integrals are path dependent or path independent. Those vector fields for which *all* line integrals between *all* pairs of points are path independent are called **conservative vector fields**.

There are five properties of a conservative vector field (P1 to P5 below). It is impossible to check the value of every line integral over every path, but it *is* possible to use any one of these five properties (particularly property P3 below) to determine whether or not a vector field is conservative. These properties are also used to simplify calculations with conservative vector fields over non-closed paths.

P1 The line integral $\displaystyle\int_A^B \underline{F}\cdot \underline{dr}$ depends only on the end points A and B and is independent of the actual path taken.

P2 The line integral around any closed curve is zero. That is $\displaystyle\oint_C \underline{F}\cdot\underline{dr} = 0$ for all C.

P3 The curl of a conservative vector field $\underline{F}$ is zero i.e. $\underline{\nabla}\times\underline{F} = \underline{0}$.

P4 For any conservative vector field $\underline{F}$, it is possible to find a scalar field ϕ such that $\underline{\nabla}\phi = \underline{F}$. Then, $\displaystyle\oint_C \underline{F}\cdot\underline{dr} = \phi(B)-\phi(A)$ where A and B are the start and end points of contour C.
[This is sometimes called the Fundamental Theorem of Line Integrals and is comparable with the Fundamental Theorem of Calculus.]

P5 All gradient fields are conservative. That is, $\underline{F} = \underline{\nabla}\phi$ is a conservative vector field for any scalar field ϕ.

Example 11

Consider the following vector fields.

1. $\underline{F}_1 = y^2\underline{i} + xy\underline{j}$ (Example 6) 2. $\underline{F}_2 = 2x\underline{i} + 2y\underline{j}$ (Example 10)

3. $\underline{F}_3 = y^2z^3\underline{i} + 2xyz^3\underline{j} + 3xy^2z^2\underline{k}$ (Example 8)

4. $\underline{F}_4 = x\underline{i} + (4x-y)\,\underline{j}$ (Task on page 14)

Determine which of these vector fields are conservative where possible by referring to the answers given in the solution. For those that are conservative find a scalar field ϕ such that $\underline{F} = \underline{\nabla}\phi$ and use property P4 to verify the values of the line integrals.

Solution

1. Two different values were obtained for line integrals over the paths C_1 and C_2. Hence, by P1, $\underline{F}_1$ is not conservative. [It is also possible to reach this conclusion from P3 by finding that $\underline{\nabla}\times\underline{F} = -y\underline{k} \neq \underline{0}$.]

Solution (contd.)

2. For the closed path consisting of C_2 and C_3 from $(0,0)$ to $(4,2)$ and back to $(0,0)$ along C_1 we obtain the value $20 + (-20) = 0$. This alone does not mean that $\underline{F}_2$ is conservative as there could be other paths giving different values. So by using P3

$$\underline{\nabla} \times \underline{F}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x & 2y & 0 \end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(0-0) = \underline{0}$$

As $\underline{\nabla} \times \underline{F}_2 = \underline{0}$, P3 gives that $\underline{F}_2$ is a conservative vector field.
Now, find a ϕ such that $\underline{F}_2 = \underline{\nabla}\phi$. Then $\dfrac{\partial \phi}{\partial x}\underline{i} + \dfrac{\partial \phi}{\partial y}\underline{j} = 2x\underline{i} + 2y\underline{j}$.

Thus

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= 2x \quad \Rightarrow \quad \phi = x^2 + f(y) \\ \frac{\partial \phi}{\partial y} &= 2y \quad \Rightarrow \quad \phi = y^2 + g(x) \end{aligned}\right\} \Rightarrow \phi = x^2 + y^2 (+\text{ constant})$$

Using P4: $\displaystyle\int_{(0,0)}^{(4,2)} \underline{F}_2 \cdot \underline{dr} = \int_{(0,0)}^{(4,2)} (\underline{\nabla}\phi) \cdot \underline{dr} = \phi(4,2) - \phi(0,0) = (4^2 + 2^2) - (0^2 + 0^2) = 20.$

3. The fact that line integrals along two different paths between the same start and end points have the same value is consistent with $\underline{F}_3$ being a conservative field according to P1. So too is the fact that the integral around a closed path is zero according to P2. However, neither fact can be used to *conclude* that $\underline{F}_3$ is a conservative field. This can be done by showing that $\underline{\nabla} \times \underline{F}_3 = \underline{0}$.

Now, $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2)\underline{i} - (3y^2z^2 - 3y^2z^2)\underline{j} + (2yz^3 - 2yz^3)\underline{k} = \underline{0}.$

As $\underline{\nabla} \times \underline{F}_3 = \underline{0}$, P3 gives that $\underline{F}_3$ is a conservative field.
To find ϕ that satisfies $\underline{\nabla}\phi = \underline{F}_3$, it is necessary to satisfy

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= y^2z^3 & &\rightarrow \quad \phi = xy^2z^3 + f(y,z) \\ \frac{\partial \phi}{\partial y} &= 2xyz^3 & &\rightarrow \quad \phi = xy^2z^3 + g(x,z) \\ \frac{\partial \phi}{\partial z} &= 3xy^2z^2 & &\rightarrow \quad \phi = xy^2z^3 + h(x,y) \end{aligned}\right\} \rightarrow \phi = xy^2z^3$$

Using P4: $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \underline{F}_3 \cdot \underline{dr} = \phi(1,1,1) - \phi(0,0,0) = 1 - 0 = 1$ in agreement with Example 8(a).

Solution (contd.)

4. As the integral along C_1 is 2 and the integral along C_2 (same start and end points but different intermediate points) is $\frac{4}{3}$, F_4 is **not** a conservative field using P1.
Note that $\underline{\nabla} \times \underline{F_4} = 4\underline{k} \neq \underline{0}$ so, using P3, this is an independent conclusion that $\underline{F_4}$ is **not** conservative.

Engineering Example 1

Work done moving a charge in an electric field

Introduction

If a charge, q, is moved through an electric field, $\underline{E}$, from A to B, then the work required is given by the line integral

$$W_{AB} = -q\int_A^B \underline{E} \cdot \underline{dr}$$

Problem in words

Compare the work done in moving a charge through the electric field around a point charge in a vacuum via two different paths.

Mathematical statement of problem

An electric field $\underline{E}$ is given by

$$\begin{aligned}\underline{E} &= \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\underline{r}} \\ &= \frac{Q}{4\pi\varepsilon_0(x^2+y^2+z^2)} \times \frac{x\underline{i}+y\underline{j}+z\underline{k}}{\sqrt{x^2+y^2+z^2}} \\ &= \frac{Q(x\underline{i}+y\underline{j}+z\underline{k})}{4\pi\varepsilon_0(x^2+y^2+z^2)^{\frac{3}{2}}}\end{aligned}$$

where $\underline{r}$ is the position vector with magnitude r and unit vector $\hat{\underline{r}}$, and $\dfrac{1}{4\pi\epsilon_0}$ is a combination of constants of proportionality, where $\epsilon_0 = 10^{-9}/36\pi\ \mathrm{F\ m^{-1}}$.

Given that $Q = 10^{-8}$C, find the work done in bringing a charge of $q = 10^{-10}$C from the point $A = (10, 10, 0)$ to the point $B = (1, 1, 0)$ (where the dimensions are in metres)

(a) by the direct straight line $y = x,\ z = 0$

(b) by the straight line pair via $C = (10, 1, 0)$

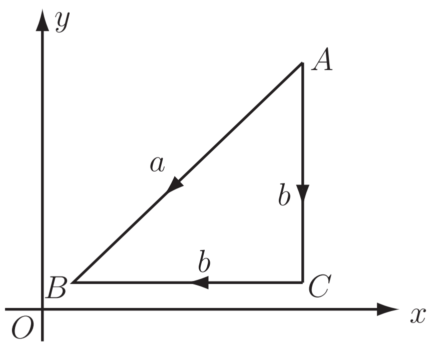

Figure 4: Two routes (a and b) along which a charge can move through an electric field

The path comprises two straight lines from $A = (10, 10, 0)$ to $B = (1, 1, 0)$ via $C = (10, 1, 0)$ (see Figure 4).

Mathematical analysis

(a) Here $Q/(4\pi\varepsilon_0) = 90$ so

$$\underline{E} = \frac{90[x\underline{i} + y\underline{j}]}{(x^2 + y^2)^{\frac{3}{2}}}$$

as $z = 0$ over the region of interest. The work done

$$\begin{aligned} W_{AB} &= -q \int_A^B \underline{E} \cdot \underline{dr} \\ &= -10^{-10} \int_A^B \frac{90}{(x^2 + y^2)^{\frac{3}{2}}} \, [x\underline{i} + y\underline{j}] \cdot [dx\underline{i} + dy\underline{j}] \end{aligned}$$

Using $y = x, \;\; dy = dx$

$$\begin{aligned} W_{AB} &= -10^{-10} \int_{x=10}^{1} \frac{90}{(2x^2)^{\frac{3}{2}}} \;\; \{x\; dx + x\; dx\} \\ &= -10^{-10} \int_{10}^{1} \frac{90}{(2\sqrt{2})} \;\; x^{-3}\; 2x\; dx \\ &= \frac{90 \times -10^{-10}}{\sqrt{2}} \int_{10}^{1} x^{-2}\; dx \\ &= \frac{9 \times -10^{-9}}{\sqrt{2}} \Big[-x^{-1} \Big]_{10}^{1} \\ &= \frac{9 \times 10^{-9}}{\sqrt{2}} \Big[x^{-1} \Big]_{10}^{1} \\ &= \frac{9 \times 10^{-9}}{\sqrt{2}} [1 - 0.1] \\ &= 5.73 \times 10^{-9} \text{ J} \end{aligned}$$

(b) The first part of the path is A to C where $x = 10$, $dx = 0$ and y goes from 10 to 1.

$$\begin{aligned}
W_{AC} &= -q \int_A^C \underline{E} \cdot \underline{dr} \\
&= -10^{-10} \int_{y=10}^{1} \frac{90}{(100+y^2)^{\frac{3}{2}}} \; [x\underline{i} + y\underline{j}] \cdot [0\underline{i} + dy\underline{j}] \\
&= -10^{-10} \int_{10}^{1} \frac{90y\ dy}{(100+y^2)^{\frac{3}{2}}} \\
&= -10^{-10} \int_{u=200}^{101} \frac{45\ du}{u^{\frac{3}{2}}} \quad \text{(substituting} \quad u = 100 + y^2,\ du = 2y\ dy) \\
&= -45 \times 10^{-10} \int_{200}^{101} u^{-\frac{3}{2}}\ du \\
&= -45 \times 10^{-10} \left[-2u^{-\frac{1}{2}} \right]_{200}^{101} \\
&= 45 \times 10^{-10} \left(\frac{2}{\sqrt{101}} - \frac{2}{\sqrt{200}} \right) = 2.59 \times 10^{-10}\text{J}
\end{aligned}$$

The second part is C to B, where $y = 1$, $dy = 0$ and x goes from 10 to 1.

$$\begin{aligned}
W_{CB} &= -10^{-10} \int_{x=10}^{1} \frac{90}{(x^2+1)^{\frac{3}{2}}} \; [x\underline{i} + y\underline{j}] \cdot [dx\underline{i} + 0\underline{j}] \\
&= -10^{-10} \int_{10}^{1} \frac{90x\ dx}{(x^2+1)^{\frac{3}{2}}} \\
&= -10^{-10} \int_{u=101}^{2} \frac{45\ du}{u^{\frac{3}{2}}} \quad \text{(substituting} \quad u = x^2 + 1,\ du = 2x\ dx) \\
&= -45 \times 10^{-10} \int_{101}^{2} u^{-\frac{3}{2}}\ du \\
&= -45 \times 10^{-10} \left[-2u^{-\frac{1}{2}} \right]_{101}^{2} \\
&= 45 \times 10^{-10} \left(\frac{2}{\sqrt{2}} - \frac{2}{\sqrt{101}} \right) = 5.468 \times 10^{-9}\text{J}
\end{aligned}$$

The sum of the two components W_{AC} and W_{CB} is 5.73×10^{-9}J.

Therefore the work done over the two paths (a) and (b) is identical.

Interpretation

In fact, the work done is independent of the route taken as the electric field $\underline{E}$ around a point charge in a vacuum is a **conservative** field.

Example 12

1. Show that $I = \int_{(0,0)}^{(2,1)} \{(2xy+1)dx + (x^2-2y)dy\}$ is independent of the path taken.
2. Find I using property P1. (Page 19)
3. Find I using property P4. (Page 19)
4. Find $I = \oint_C \{(2xy+1)dx + (x^2-2y)dy\}$ where C is
   (a) the circle $x^2+y^2=1$
   (b) the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Solution

1. The integral $I = \int_{(0,0)}^{(2,1)} \{(2xy+1)dx + (x^2-2y)dy\}$ may be re-written $\int_C \underline{F}\cdot\underline{dr}$ where $\underline{F} = (2xy+1)\underline{i} + (x^2-2y)\underline{j}$.

$$\text{Now } \underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+1 & x^2-2y & 0 \end{vmatrix} = 0\underline{i}+0\underline{j}+0\underline{k} = \underline{0}$$

As $\underline{\nabla}\times\underline{F} = \underline{0}$, $\underline{F}$ is a conservative field and I is independent of the path taken between $(0,0)$ and $(2,1)$.

2. As I is independent of the path taken from $(0,0)$ to $(2,1)$, it can be evaluated along **any** such path. One possibility is the straight line $y = \frac{1}{2}x$. On this line, $dy = \frac{1}{2}dx$. The integral I becomes

$$\begin{aligned} I &= \int_{(0,0)}^{(2,1)} \{(2xy+1)dx + (x^2-2y)dy\} \\ &= \int_{x=0}^{2} \left\{(2x\times\frac{1}{2}x+1)dx + (x^2-x)\frac{1}{2}dx\right\} \\ &= \int_0^2 (\frac{3}{2}x^2 - \frac{1}{2}x+1)dx \\ &= \left[\frac{1}{2}x^3 - \frac{1}{4}x^2 + x\right]_0^2 = 4-1+2-0 = 5 \end{aligned}$$

Solution (contd.)

3. If $\underline{F} = \underline{\nabla}\phi$ then

$$\left.\begin{array}{l}\dfrac{\partial \phi}{\partial x} = 2xy + 1 \quad \rightarrow \quad \phi = x^2y + x + f(y) \\[2ex] \dfrac{\partial \phi}{\partial y} = x^2 - 2y \quad \rightarrow \quad \phi = x^2y - y^2 + g(x)\end{array}\right\} \rightarrow \phi = x^2y + x - y^2 + C.$$

These are consistent if $\phi = x^2y + x - y^2$ (plus a constant which may be omitted since it cancels).
So $I = \phi(2,1) - \phi(0,0) = (4 + 2 - 1) - 0 = 5$

4. As $\underline{F}$ is a conservative field, all integrals around a closed contour are zero.

Exercises

1. Determine whether the following vector fields are conservative

   (a) $\underline{F} = (x - y)\underline{i} + (x + y)\underline{j}$

   (b) $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$

   (c) $\underline{F} = 2x\underline{i} + (xz - 2)\underline{j} + xy\underline{k}$

   (d) $\underline{F} = x^2z\underline{i} + y^2z\underline{j} + \frac{1}{3}(x^3 + y^3)\underline{k}$

2. Consider the integral $\displaystyle\int_C \underline{F} \cdot \underline{dr}$ with $\underline{F} = 3x^2y^2\underline{i} + (2x^3y - 1)\underline{j}$. From Exercise 1(b) $\underline{F}$ is a conservative vector field. Find a scalar field ϕ so that $\underline{\nabla}\phi = \underline{F}$. Use property P4 to evaluate the integral $\displaystyle\int_C \underline{F} \cdot \underline{dr}$ where C is an integral with start-point $(0, 0)$ and end point $(1, 4)$.

3. For the following conservative vector fields $\underline{F}$, find a scalar field ϕ such that $\underline{\nabla}\phi = \underline{F}$ and hence evaluate the $I = \displaystyle\int_C \underline{F} \cdot \underline{dr}$ for the contours C indicated.

   (a) $\underline{F} = (4x^3y - 2x)\underline{i} + (x^4 - 2y)\underline{j}$; any path from $(0, 0)$ to $(2, 1)$.

   (b) $\underline{F} = (e^x + y^3)\underline{i} + (3xy^2)\underline{j}$; closed path starting from any point on the circle $x^2 + y^2 = 1$.

   (c) $\underline{F} = (y^2 + \sin z)\underline{i} + 2xy\underline{j} + x\cos z\underline{k}$; any path from $(1, 1, 0)$ to $(2, 0, \pi)$.

   (d) $\underline{F} = \dfrac{1}{x}\underline{i} + 4y^3z^2\underline{j} + 2y^4z\underline{k}$; any path from $(1, 1, 1)$ to $(1, 2, 3)$.

Answers

1. (a) No, (b) Yes, (c) No, (d) Yes

2. $x^3y^2 - y + C$, $\quad 12$

3. (a) $x^4y - x^2 - y^2$, 11; (b) $e^x + xy^3$, 0; (c) $xy^2 + x\sin z$, -1; (d) $\ln x + y^4z^2$,143

4. Vector line integrals

It is also possible to form less commonly used integrals of the types:

$$\int_C f(x,y,z)\,\underline{dr} \quad \text{and} \quad \int_C \underline{F}(x,y,z)\times\underline{dr}.$$

Each of these integrals evaluates to a vector.

Remembering that $\underline{dr} = dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}$, an integral of the form $\int_C f(x,y,z)\,\underline{dr}$ becomes $\int_C f(x,y,z)dx\ \ \underline{i} + \int_C f(x,y,z)\ \ dy\ \ \underline{j} + \int_C f(x,y,z)dz\ \ \underline{k}$. The first term can be evaluated by expressing y and z in terms of x. Similarly the second and third terms can be evaluated by expressing all terms as functions of y and z respectively. Alternatively, all variables can be expressed in terms of a parameter t. If an integral is two-dimensional, the term in z will be absent.

Example 13

Evaluate the integral $\int_C xy^2\underline{dr}$ where C represents the contour $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution

This is a two-dimensional integral so the term in z will be absent.

$$\begin{aligned}
I &= \int_C xy^2\underline{dr} \\
&= \int_C xy^2(dx\underline{i} + dy\underline{j}) \\
&= \int_C xy^2dx\ \underline{i} + \int_C xy^2\ dy\ \underline{j} \\
&= \int_{x=0}^{1} x(x^2)^2dx\ \underline{i} + \int_{y=0}^{1} y^{1/2}y^2\ dy\ \underline{j} \\
&= \int_0^1 x^5dx\ \underline{i} + \int_0^1 y^{5/2}\ dy\ \underline{j} \\
&= \left[\frac{1}{6}x^6\right]_0^1 \underline{i} + \left[\frac{2}{7}x^{7/2}\right]_0^1 \underline{j} \\
&= \frac{1}{6}\underline{i} + \frac{2}{7}\underline{j}
\end{aligned}$$

Example 14

Find $I = \int_C x\underline{dr}$ for the contour C given parametrically by $x = \cos t$, $y = \sin t$, $z = t - \pi$ starting at $t = 0$ and going to $t = 2\pi$, i.e. the contour starts at $(1, 0, -\pi)$ and finishes at $(1, 0, \pi)$.

Solution

The integral becomes $\int_C x(dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k})$.

Now, $x = \cos t$, $y = \sin t$, $z = t - \pi$ so $dx = -\sin t\,dt$, $dy = \cos t\,dt$ and $dz = dt$. So

$$
\begin{aligned}
I &= \int_0^{2\pi} \cos t(-\sin t\,dt\,\underline{i} + \cos t\,dt\,\underline{j} + dt\,\underline{k}) \\
&= -\int_0^{2\pi} \cos t \sin t\,dt\,\underline{i} + \int_0^{2\pi} \cos^2 t\,dt\,\underline{j} + \int_0^{2\pi} \cos t\,dt\,\underline{k} \\
&= -\frac{1}{2}\int_0^{2\pi} \sin 2t\,dt\,\underline{i} + \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt\,\underline{j} + \Big[\sin t\Big]_0^{2\pi}\,\underline{k} \\
&= \frac{1}{4}\Big[\cos 2t\Big]_0^{2\pi}\,\underline{i} \;+\; \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi}\,\underline{j} \;+\; 0\underline{k} \\
&= 0\underline{i} + \pi\,\underline{j} = \pi\underline{j}
\end{aligned}
$$

Integrals of the form $\int_C \underline{F} \times \underline{dr}$ can be evaluated as follows. If the vector field $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$ and $\underline{dr} = dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}$ then:

$$
\begin{aligned}
\underline{F} \times \underline{dr} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ F_1 & F_2 & F_3 \\ dx & dy & dz \end{vmatrix} &= (F_2\,dz - F_3\,dy)\underline{i} + (F_3\,dx - F_1\,dz)\underline{j} + (F_1\,dy - F_2\,dx)\underline{k} \\
&= (F_3\underline{j} - F_2\underline{k})dx + (F_1\underline{k} - F_3\underline{i})dy + (F_2\underline{i} - F_1\underline{j})dz
\end{aligned}
$$

There are thus a maximum of six terms involved in one such integral; the exact details may dictate which method to use.

Example 15

Evaluate the integral $\int_C (x^2\underline{i} + 3xy\underline{j}) \times \underline{dr}$ where C represents the curve $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution

Note that the z components of both $\underline{F}$ and $\underline{dr}$ are zero.

$$\underline{F} \times \underline{dr} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x^2 & 3xy & 0 \\ dx & dy & 0 \end{vmatrix} = (x^2dy - 3xydx)\underline{k} \quad \text{and}$$

$$\int_C (x^2\underline{i} + 3xy\underline{j}) \times \underline{dr} = \int_C (x^2dy - 3xydx)\underline{k}$$

Now, on C, $\quad y = 2x^2 \quad dy = 4xdx \quad$ and

$$\begin{aligned} \int_C (x^2\underline{i} + 3xy\underline{j}) \times \underline{dr} &= \int_C \{x^2dy - 3xydx\}\underline{k} \\ &= \int_{x=0}^{1} \left\{x^2 \times 4xdx - 3x \times 2x^2dx\right\} \underline{k} \\ &= \int_0^1 -2x^3dx\underline{k} \\ &= -\left[\frac{1}{2}x^4\right]_0^1 \underline{k} \\ &= -\frac{1}{2}\underline{k} \end{aligned}$$

Engineering Example 2

Force on a loop due to a magnetic field

Introduction

A current $\underline{I}$ in a magnetic field $\underline{B}$ is subject to a force $\underline{F}$ given by

$$\underline{F} = I\ \underline{dr} \times \underline{B}$$

where the current can be regarded as having magnitude I and flowing (positive charge) in the direction given by the vector $\underline{dr}$. The force is known as the Lorentz force and is responsible for the workings of an electric motor. If current flows around a loop, the total force on the loop is given by the integral of $\underline{F}$ around the loop, i.e.

$$\underline{F} = \oint (I\ \underline{dr} \times \underline{B}) = -I \oint (\underline{B} \times \underline{dr})$$

where the closed path of the integral represents one circuit of the loop.

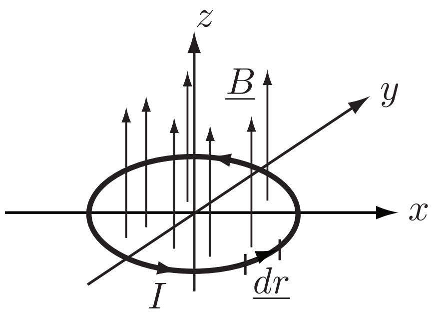

Figure 5: The magnetic field through a loop of current

Problem in words

A current of 1 amp flows around a circuit in the shape of the unit circle in the Oxy plane. A magnetic field of 1 tesla (T) in the positive z-direction is present. Find the total force on the circuit loop.

Mathematical statement of problem

Choose an origin at the centre of the circuit and use polar coordinates to describe the position of any point on the circuit and the length of a small element.

Calculate the line integral around the circuit to give the force required using the given values of current and magnetic field.

Mathematical analysis

The circuit is described parametrically by

$$x = \cos\theta \quad y = \sin\theta \quad z = 0$$

with

$$\begin{aligned} \underline{dr} &= -\sin\theta\ d\theta\ \underline{i} + \cos\theta\ d\theta\ \underline{j} \\ \underline{B} &= B\ \underline{k} \end{aligned}$$

since $\underline{B}$ is constant. Therefore, the force on the circuit is given by

$$\underline{F} = -IB \oint \underline{k} \times \underline{dr} = -\oint \underline{k} \times \underline{dr} \qquad \text{(since } I = 1 \text{ A and } B = 1 \text{ T)}$$

where

$$\begin{aligned} \underline{k} \times \underline{dr} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1 \\ -\sin\theta\, d\theta & \cos\theta\, d\theta & 0 \end{vmatrix} \\ &= \left(-\cos\theta\, \underline{i} - \sin\theta\, \underline{j}\,\right) d\theta \end{aligned}$$

So

$$\begin{aligned} \underline{F} &= -\int_{\theta=0}^{2\pi} \left(-\cos\theta\, \underline{i} - \sin\theta\, \underline{j}\,\right) d\theta \\ &= \Big[\sin\theta\, \underline{i} - \cos\theta\, \underline{j}\,\Big]_{\theta=0}^{2\pi} \\ &= (0-0)\, \underline{i} - (1-1)\, \underline{j} = \underline{0} \end{aligned}$$

Hence there is no net force on the loop.

Interpretation

At any given point of the circle, the force on the point opposite is of the same magnitude but opposite direction, and so cancels, leaving a zero net force.

Tip: Use symmetry arguments to avoid detailed calculations whenever possible!

A scalar or vector involved in a vector line integral may itself be a vector derivative as this next Example illustrates.

Example 16

Find the vector line integral $\int_C (\underline{\nabla} \cdot \underline{F})\, \underline{dr}$ where $\underline{F}$ is the vector $x^2\underline{i} + 2xy\underline{j} + 2xz\underline{k}$ and C is the curve $y = x^2$, $z = x^3$ from $x = 0$ to $x = 1$ i.e. from $(0,0,0)$ to $(1,1,1)$. Here $\underline{\nabla} \cdot \underline{F}$ is the (scalar) divergence of the vector $\underline{F}$.

Solution

As $\underline{F} = x^2\underline{i} + 2xy\underline{j} + 2xz\underline{k}$, $\underline{\nabla} \cdot \underline{F} = 2x + 2x + 2x = 6x$.
The integral

$$\begin{aligned}\int_C (\underline{\nabla} \cdot \underline{F})\, \underline{dr} &= \int_C 6x(dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}) \\ &= \int_C 6x\,dx\,\underline{i} + \int_C 6x\,dy\,\underline{j} + \int_C 6x\,dz\,\underline{k}\end{aligned}$$

The first term is

$$\int_C 6x\,dx\,\underline{i} = \int_{x=0}^{1} 6x\,dx\,\underline{i} = \Big[3x^2\Big]_0^1 \underline{i} = 3\underline{i}$$

In the second term, as $y = x^2$ on C, dy may be replaced by $2x\,dx$ so

$$\int_C 6x\,dy\,\underline{j} = \int_{x=0}^{1} 6x \times 2x\,dx\,\underline{j} = \int_0^1 12x^2\,dx\,\underline{j} = \Big[4x^3\Big]_0^1 \underline{j} = 4\underline{j}$$

In the third term, as $z = x^3$ on C, dz may be replaced by $3x^2\,dx$ so

$$\int_C 6x\,dz\,\underline{k} = \int_{x=0}^{1} 6x \times 3x^2\,dx\,\underline{k} = \int_0^1 18x^3\,dx\,\underline{k} = \left[\frac{9}{2}x^4\right]_0^1 \underline{k} = \frac{9}{2}\underline{k}$$

On summing, $\int_C (\underline{\nabla} \cdot \underline{F})\, \underline{dr} = 3\underline{i} + 4\underline{j} + \dfrac{9}{2}\underline{k}$.

Task

Find the vector line integral $\int_C f\underline{dr}$ where $f = x^2$ and C is

(a) the curve $y = x^{1/2}$ from $(0,0)$ to $(9,3)$.

(b) the line $y = x/3$ from $(0,0)$ to $(9,3)$.

Your solution

Answer

(a) $\displaystyle\int_0^9 (x^2\underline{i} + \frac{1}{2}x^{3/2}\underline{j})dx = 243\underline{i} + \frac{243}{5}\underline{j},$ (b) $\displaystyle\int_0^9 (x^2\underline{i} + \frac{1}{3}x^2\underline{j})dx = 243\underline{i} + 81\underline{j}.$

Task

Evaluate the vector line integral $\displaystyle\int_C \underline{F} \times \underline{dr}$ when C represents the contour $y = 4 - 4x$, $z = 2 - 2x$ from $(0, 4, 2)$ to $(1, 0, 0)$ and $\underline{F}$ is the vector field $(x - z)\underline{j}$.

Your solution

Answer

$$\int_0^1 \{(4 - 6x)\underline{i} + (2 - 3x)\underline{k}\} = \underline{i} + \frac{1}{2}\underline{k}$$

Exercises

1. Evaluate the vector line integral $\int_C (\underline{\nabla} \cdot \underline{F})\, \underline{dr}$ in the case where $\underline{F} = x\underline{i} + xy\underline{j} + xy^2\underline{k}$ and C is the contour described by $x = 2t$, $y = t^2$, $z = 1 - t$ for t starting at $t = 0$ and going to $t = 1$.

2. When C is the contour $y = x^3$, $z = 0$, from $(0, 0, 0)$ to $(1, 1, 0)$, evaluate the vector line integrals

   (a) $\int_C \{\underline{\nabla}(xy)\} \times \underline{dr}$

   (b) $\int_C \left\{\underline{\nabla} \times (x^2\underline{i} + y^2\underline{k})\right\} \times \underline{dr}$

Answers

1. $\int_C (1 + x)(dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}) = 4\underline{i} + \frac{7}{3}\underline{j} - 2\underline{k}$,

2. (a) $\underline{k}\int_C y\,dy - x\,dx = 0\underline{k} = \underline{0}$, (b) $\underline{k}\int_C 2y\,dy = 1\underline{k} = \underline{k}$

Surface and Volume Integrals

29.2

Introduction

A vector or scalar field - including one formed from a vector derivative (div, grad or curl) - can be integrated over a surface or volume. This Section shows how to carry out such operations.

Prerequisites

Before starting this Section you should ...

- be familiar with vector derivatives
- be familiar with double and triple integrals

Learning Outcomes

On completion you should be able to ...

- carry out operations involving integration of scalar and vector fields

1. Surface integrals involving vectors

The unit normal

For the surface of any three-dimensional shape, it is possible to find a vector lying perpendicular to the surface and with magnitude 1. The unit vector points outwards from a closed surface and is usually denoted by $\hat{\underline{n}}$.

Example 17

If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ find the unit normal $\hat{\underline{n}}$.

Solution

The unit normal at the point $P(x, y, z)$ points away from the centre of the sphere i.e. it lies in the direction of $x\underline{i} + y\underline{j} + z\underline{k}$. To make this a unit vector it must be divided by its magnitude $\sqrt{x^2 + y^2 + z^2}$ i.e. the unit vector is

$$\begin{aligned} \hat{\underline{n}} &= \frac{x}{\sqrt{x^2+y^2+z^2}}\underline{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\underline{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\underline{k} \\ &= \frac{x}{a}\underline{i} + \frac{y}{a}\underline{j} + \frac{z}{a}\underline{k} \end{aligned}$$

where $a = \sqrt{x^2 + y^2 + z^2}$ is the radius of the sphere.

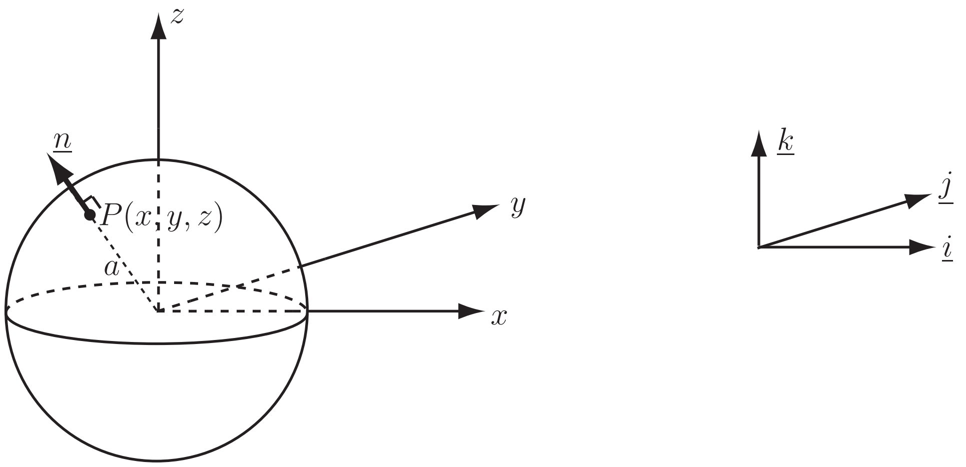

Figure 6: A unit normal $\hat{\underline{n}}$ to a sphere

Example 18

For the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, find the unit outward normal $\underline{\hat{n}}$ for each face.

Solution

On the face given by $x = 0$, the unit normal points in the negative x-direction. Hence the unit normal is $-\underline{i}$. Similarly :-

On the face $x = 1$ the unit normal is $\underline{i}$. On the face $y = 0$ the unit normal is $-\underline{j}$.

On the face $y = 1$ the unit normal is $\underline{j}$. On the face $z = 0$ the unit normal is $-\underline{k}$.

On the face $z = 1$ the unit normal is $\underline{k}$.

$\underline{dS}$ and the unit normal

The vector $\underline{dS}$ is a vector, being an element of the surface with magnitude $du\, dv$ and direction perpendicular to the surface.

If the plane in question is the Oxy plane, then $\underline{dS} = \underline{\hat{n}}\, du\, dv = \underline{k}\, dx\, dy$.

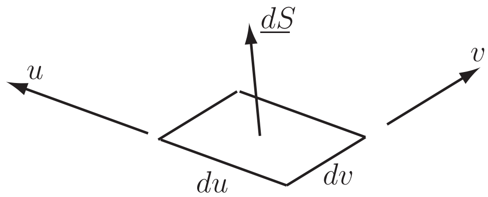

Figure 7: The vector $\underline{dS}$ as an element of a surface, with magnitude $du\, dv$

If the plane in question is not one of the three coordinate planes (Oxy, Oxz, Oyz), appropriate adjustments must be made to express $\underline{dS}$ in terms of two of dx and dy and dz.

Example 19

The rectangle $OABC$ lies in the plane $z = y$ (Figure 8).

The vertices are $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 1, 1)$.

Find a unit vector $\underline{\hat{n}}$ normal to the plane and an appropriate vector $\underline{dS}$ expressed in terms of dx and dy.

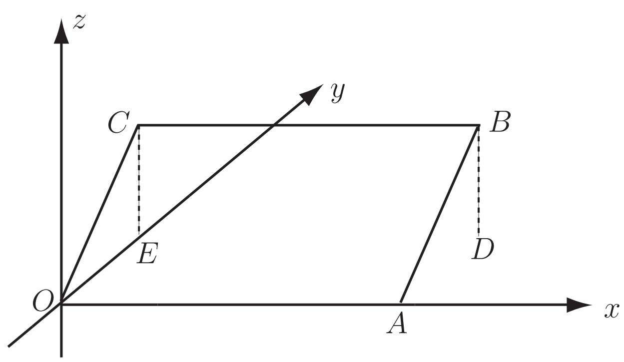

Figure 8: The plane $z = y$ passing through $OABC$

Solution

Note that two vectors in the rectangle are $\overrightarrow{OA} = \underline{i}$ and $\overrightarrow{OC} = \underline{j} + \underline{k}$. A vector perpendicular to the plane is $\underline{i} \times (\underline{j} + \underline{k}) = -\underline{j} + \underline{k}$. However, this vector is of magnitude $\sqrt{2}$ so the unit normal vector is $\underline{\hat{n}} = \dfrac{1}{\sqrt{2}}(-\underline{j} + \underline{k}) = -\dfrac{1}{\sqrt{2}}\underline{j} + \dfrac{1}{\sqrt{2}}\underline{k}$.

The vector $\underline{dS}$ is therefore $(-\dfrac{1}{\sqrt{2}}\underline{j} + \dfrac{1}{\sqrt{2}}\underline{k})\, du\, dv$ where du and dv are increments in the plane of the rectangle $OABC$. Now, one increment, say du, may point in the x-direction while dv will point in a direction up the plane, parallel to OC. Thus $du = dx$ and (by Pythagoras) $dv = \sqrt{(dy)^2 + (dz)^2}$. However, as $z = y$, $dz = dy$ and hence $dv = \sqrt{2}dy$.
Thus, $\underline{dS} = (-\dfrac{1}{\sqrt{2}}\underline{j} + \dfrac{1}{\sqrt{2}}\underline{k})\, dx\, \sqrt{2}\, dy = (-\underline{j} + \underline{k})\, dx\, dy$.

Note :- the factor of $\sqrt{2}$ could also have been found by comparing the area of rectangle $OABC$, i.e. 1, with the area of its projection in the Oxy plane i.e. $OADE$ with area $\dfrac{1}{\sqrt{2}}$.

Integrating a scalar field

A function can be integrated over a surface by constructing a double integral and integrating in a manner similar to that shown in HELM 27.1 and HELM 27.2. Often, such integrals can be carried out with respect to an element containing the unit normal.

Example 20

Evaluate the integral

$$\int_A \frac{1}{1+x^2}\, \underline{dS}$$

over the area A where A is the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$.

Solution

In this integral, $\underline{dS}$ becomes $\underline{k}\, dx\, dy$ i.e. the unit normal times the surface element. Thus the integral is

$$\begin{aligned}
\int_{y=0}^{1}\int_{x=0}^{1} \frac{\underline{k}}{1+x^2}\, dx\, dy &= \underline{k}\int_{y=0}^{1} \Big[\tan^{-1} x\Big]_0^1 dy \\
&= \underline{k}\int_{y=0}^{1} \Big[(\frac{\pi}{4} - 0)\Big]_0^1 dy = \frac{\pi}{4}\underline{k}\int_{y=0}^{1} dy \\
&= \frac{\pi}{4}\underline{k}
\end{aligned}$$

Example 21

Find $\displaystyle\iint_S u\,\underline{dS}$ where $u = x^2 + y^2 + z^2$ and S is the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

The unit cube has six faces and the unit normal vector $\underline{\hat{n}}$ points in a different direction on each face; see Example 18. The surface integral must be evaluated for each face separately and the results summed.

On the face $x = 0$, the unit normal $\underline{\hat{n}} = -\underline{i}$ and the surface integral is

$$\begin{aligned}\int_{y=0}^{1}\int_{z=0}^{1}(0^2+y^2+z^2)(-\underline{i})dzdy &= -\underline{i}\int_{y=0}^{1}\left[y^2z+\frac{1}{3}z^3\right]_{z=0}^{1}dy \\ &= -\underline{i}\int_{y=0}^{1}\left(y^2+\frac{1}{3}\right)dy = -\underline{i}\left[\frac{1}{3}y^3+\frac{1}{3}y\right]_0^1 = -\frac{2}{3}\underline{i}\end{aligned}$$

On the face $x = 1$, the unit normal $\underline{\hat{n}} = \underline{i}$ and the surface integral is

$$\begin{aligned}\int_{y=0}^{1}\int_{z=0}^{1}(1^2+y^2+z^2)(\underline{i})dzdy &= \underline{i}\int_{y=0}^{1}\left[z+y^2z+\frac{1}{3}z^3\right]_{z=0}^{1}dy \\ &= \underline{i}\int_{y=0}^{1}\left(y^2+\frac{4}{3}\right)dy = \underline{i}\left[\frac{1}{3}y^3+\frac{4}{3}y\right]_0^1 = \frac{5}{3}\underline{i}\end{aligned}$$

The net contribution from the faces $x = 0$ and $x = 1$ is $-\frac{2}{3}\underline{i} + \frac{5}{3}\underline{i} = \underline{i}$.

Due to the symmetry of the scalar field u and the unit cube, the net contribution from the faces $y = 0$ and $y = 1$ is $\underline{j}$ while the net contribution from the faces $z = 0$ and $z = 1$ is $\underline{k}$.

Adding, we obtain $\displaystyle\iint_S u\underline{dS} = \underline{i} + \underline{j} + \underline{k}$

Key Point 4

A scalar function integrated with respect to a normal vector $\underline{dS}$ gives a vector quantity.

When the surface does not lie in one of the planes Oxy, Oxz, Oyz, extra care must be taken when finding $\underline{dS}$.

Example 22

Find $\int\int_S (\underline{\nabla} \cdot \underline{F}) \underline{dS}$ where $\underline{F} = 2x\underline{i} + yz\underline{j} + xy\underline{k}$ and S is the surface of the triangle with vertices at $(0,0,0)$, $(1,0,0)$ and $(1,1,0)$.

Solution

Note that $\underline{\nabla} \cdot \underline{F} = 2 + z = 2$ as $z = 0$ everywhere along S. As the triangle lies in the Oxy plane, the normal vector $\underline{n} = \underline{k}$ and $\underline{dS} = \underline{k}dydx$.
Thus,

$$\int\int_S (\underline{\nabla} \cdot \underline{F}) \underline{dS} = \int_{x=0}^{1} \int_{y=0}^{x} 2dydx\underline{k} = \int_0^1 \Big[2y\Big]_0^x dx\underline{k} = \int_0^1 2xdx\underline{k} = \Big[x^2\Big]_0^1 \underline{k} = \underline{k}$$

Here the scalar function being integrated was the divergence of a vector function.

Example 23

Find $\int\int_S f\ \underline{dS}$ where f is the function $2x$ and S is the surface of the triangle bounded by $(0,0,0)$, $(0,1,1)$ and $(1,0,1)$. (See Figure 9.)

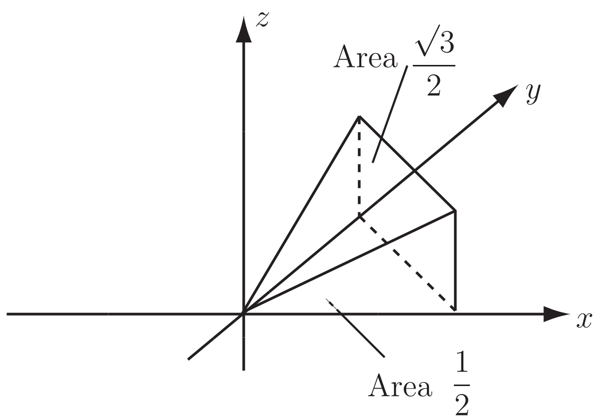

Figure 9: The triangle defining the area S

Solution

The unit vector $\underline{n}$ is perpendicular to two vectors in the plane e.g. $(\underline{j} + \underline{k})$ and $(\underline{i} + \underline{k})$. The vector $(\underline{j} + \underline{k}) \times (\underline{i} + \underline{k}) = \underline{i} + \underline{j} - \underline{k}$ which has magnitude $\sqrt{3}$. Hence the unit normal vector $\hat{\underline{n}} = \frac{1}{\sqrt{3}}\underline{i} + \frac{1}{\sqrt{3}}\underline{j} - \frac{1}{\sqrt{3}}\underline{k}$.

As the area of the triangle S is $\frac{\sqrt{3}}{2}$ and the area of its projection in the Oxy plane is $\frac{1}{2}$, the vector $\underline{dS} = \dfrac{\sqrt{3}/2}{1/2}\hat{\underline{n}}\, dydx = (\underline{i} + \underline{j} + \underline{k})dydx$.

Thus

$$\begin{aligned}
\iint_S f\underline{dS} &= (\underline{i} + \underline{j} + \underline{k}) \int_{x=0}^{1} \int_{y=0}^{1-x} 2x\, dydx \\
&= (\underline{i} + \underline{j} + \underline{k}) \int_{x=0}^{1} \Big[2xy \Big]_{y=0}^{1-x} dx \\
&= (\underline{i} + \underline{j} + \underline{k}) \int_{x=0}^{1} (2x - 2x^2) dx \\
&= (\underline{i} + \underline{j} + \underline{k}) \left[x^2 - \frac{2}{3}x^3 \right]_0^1 = \frac{1}{3}(\underline{i} + \underline{j} + \underline{k})
\end{aligned}$$

Task

Evaluate the integral $\displaystyle\iint_S 4x\ \underline{dS}$ where S represents the trapezium with vertices at $(0, 0)$, $(3, 0)$, $(2, 1)$ and $(0, 1)$.

(a) Find the vector $\underline{dS}$:

Your solution

Answer

$\underline{k}\ dx\ dy$

(b) Write the surface integral as a double integral:

Your solution

Answer

It is easier to integrate first with respect to x. This gives $\displaystyle\int_{y=0}^{1}\int_{x=0}^{3-y} 4x\ dx\, dy\, \underline{k}$.

The range of values of y is $y = 0$ to $y = 1$.

For each value of y, x varies from $x = 0$ to $x = 3 - y$

(c) Evaluate this double integral:

Your solution

Answer

$\dfrac{38}{3}\underline{k}$

Exercises

1. Evaluate the integral $\displaystyle\iint_S xy\underline{dS}$ where S is the triangle with vertices at $(0,0,4)$, $(0,2,0)$ and $(1,0,0)$.

2. Find the integral $\displaystyle\iint_S xyz\underline{dS}$ where S is the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

3. Evaluate the integral $\displaystyle\iint_S \left[\underline{\nabla} \cdot (x^2\underline{i} + yz\underline{j} + x^2y\underline{k})\right] \underline{dS}$ where S is the rectangle with vertices at $(1,0,0)$, $(1,1,0)$, $(1,1,1)$ and $(1,0,1)$.

Answers	1. $\dfrac{2}{3}\underline{i} + \dfrac{1}{3}\underline{j} + \dfrac{1}{6}\underline{k}$	2. $\dfrac{1}{4}(\underline{x} + \underline{y} + \underline{z})$,	3. $\dfrac{5}{2}\underline{i}$

Integrating a vector field

In a similar manner to the case of a scalar field, a vector field may be integrated over a surface. Two common types of integral are $\displaystyle\int_S \underline{F}(\underline{r}) \cdot \underline{dS}$ and $\displaystyle\int_S \underline{F}(\underline{r}) \times \underline{dS}$ which integrate to a scalar and a vector respectively. Again, when $\underline{dS}$ is expressed appropriately, the expression will reduce to a double integral. The form $\displaystyle\int_S \underline{F}(r) \cdot \underline{dS}$ has many important applications, e.g. the flux of a vector field such as an electric or magnetic field.

Example 24

Evaluate the integral

$$\int_A (x^2y\underline{i} + z\underline{j} + (2x+y)\underline{k}) \cdot \underline{dS}$$

over the area A where A is the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$.

Solution

On A, the unit normal is $dx\,dy\,\underline{k}$

$$\begin{aligned}
\therefore \quad & \int_A (x^2y\underline{i} + z\underline{j} + (2x+y)\underline{k}) \cdot (\underline{k}\ dx\,dy) \\
= \quad & \int_{y=0}^{1}\int_{x=0}^{1} (2x+y)\ dx\,dy = \int_{y=0}^{1}\Big[x^2 + xy\Big]_{x=0}^{1} dy \\
= \quad & \int_{y=0}^{1} (1+y)dy = \left[y + \frac{1}{2}y^2\right]_0^1 = \frac{3}{2}
\end{aligned}$$

Example 25

Evaluate $\int_A \underline{r} \cdot \underline{dS}$ where A represents the surface of the unit cube

$0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1$ and $\underline{r}$ represents the vector $x\underline{i} + y\underline{j} + z\underline{k}$.

Solution

The vector $\underline{dS}$ (in the direction of the normal vector) will be a constant vector on each face, but will be different for each face.

On the face $x = 0$, $\underline{dS} = -dy\,dz\,\underline{i}$ and the integral on this face is

$$\int_{z=0}^{1} \int_{y=0}^{1} (0\underline{i} + y\underline{j} + z\underline{k}) \cdot (-dy\,dz\,\underline{i}) = \int_{z=0}^{1} \int_{y=0}^{1} 0 \; dy\,dz = 0$$

Similarly on the face $y = 0$, $\underline{dS} = -dx\,dz\,\underline{j}$ and the integral on this face is

$$\int_{z=0}^{1} \int_{x=0}^{1} (x\underline{i} + 0\underline{j} + z\underline{k}) \cdot (-dx\,dz\,\underline{j}) = \int_{z=0}^{1} \int_{x=0}^{1} 0 \; dx\,dz = 0$$

Furthermore on the face $z = 0$, $\underline{dS} = -dx\,dy\,\underline{k}$ and the integral on this face is

$$\int_{x=0}^{1} \int_{y=0}^{1} (x\underline{i} + y\underline{j} + 0\underline{k}) \cdot (-dx\,dy\,\underline{k}) = \int_{x=0}^{1} \int_{y=0}^{1} 0 \; dx\,dy = 0$$

On these three faces, the contribution to the integral is thus zero.

However, on the face $x = 1$, $\underline{dS} = +dy\,dz\,\underline{i}$ and the integral on this face is

$$\int_{z=0}^{1} \int_{y=0}^{1} (1\underline{i} + y\underline{j} + z\underline{k}) \cdot (+dy\,dz\,\underline{i}) = \int_{z=0}^{1} \int_{y=0}^{1} 1 \; dy\,dz = 1$$

Similarly, on the face $y = 1$, $\underline{dS} = +dx\,dz\,\underline{j}$ and the integral on this face is

$$\int_{z=0}^{1} \int_{x=0}^{1} (x\underline{i} + 1\underline{j} + z\underline{k}) \cdot (+dx\,dz\,\underline{j}) = \int_{z=0}^{1} \int_{x=0}^{1} 1 \; dx\,dz = 1$$

Finally, on the face $z = 1$, $\underline{dS} = +dx\,dy\,\underline{k}$ and the integral on this face is

$$\int_{y=0}^{1} \int_{x=0}^{1} (x\underline{i} + y\underline{j} + 1\underline{k}) \cdot (+dx\,dy\,\underline{k}) = \int_{y=0}^{1} \int_{x=0}^{1} 1 \; dx\,dy = 1$$

Adding together the contributions gives $\int_A \underline{r} \cdot \underline{dS} = 0 + 0 + 0 + 1 + 1 + 1 = 3$

Engineering Example 3

Magnetic flux

Introduction

The magnetic flux through a surface is given by $\displaystyle\iint_S \underline{B}\cdot\underline{dS}$ where S is the surface under consideration, $\underline{B}$ is the magnetic field and $\underline{dS}$ is the vector normal to the surface.

Problem in words

The magnetic field generated by an infinitely long vertical wire on the z-axis, carrying a current I, is given by:

$$\underline{B} = \frac{\mu_0 I}{2\pi}\left(\frac{-y\underline{i} + x\underline{j}}{x^2+y^2}\right)$$

Find the flux through a rectangular region (with sides parallel to the axes) on the plane $y = 0$.

Mathematical statement of problem

Find the integral $\displaystyle\iint_S \underline{B}\cdot\underline{dS}$ over the surface, $x_1 \leq x \leq x_2, \quad z_1 \leq z \leq z_2$. (see Figure 10 which shows part of the plane $y = 0$ for which the flux is to be found and a single magnetic field line. The strength of the field is inversely proportional to the distance from the axis.)

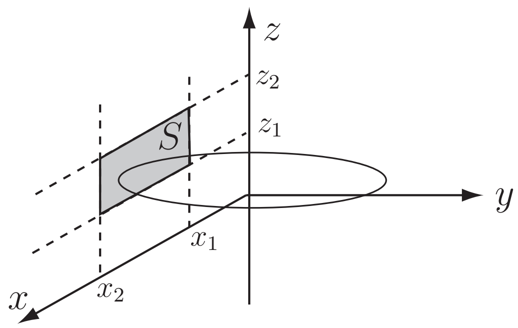

Figure 10: The surface S defined by $x_1 \leq x \leq x_2,\ z_1 \leq z \leq z_2$

Mathematical analysis

On $y = 0$, $\underline{B} = \dfrac{\mu_0 I}{2\pi x}\ \underline{j}$ and $\underline{dS} = dx\,dz\,\underline{j}$ so $\underline{B}\cdot\underline{dS} = \dfrac{\mu_0 I}{2\pi x}\,dx\,dz$

The flux is given by the double integral:

$$\begin{aligned}
\int_{z=z_1}^{z_2}\int_{x=x_1}^{x_2}\frac{\mu_0 I}{2\pi x}\,dx\,dz &= \frac{\mu_0 I}{2\pi}\int_{z=z_1}^{z_2}\Big[\ln x\Big]_{x_1}^{x_2}\,dz \\
&= \frac{\mu_0 I}{2\pi}\int_{z=z_1}^{z_2}\Big(\ln x_2 - \ln x_1\Big)\,dz \\
&= \frac{\mu_0 I}{2\pi}\Big[z\big(\ln x_2 - \ln x_1\big)\Big]_{z=z_1}^{z_2} = \frac{\mu_0 I}{2\pi}\,(z_2 - z_1)\ln\left(\frac{x_2}{x_1}\right)
\end{aligned}$$

Interpretation

The magnetic flux increases in direct proportion to the extent of the side parallel to the axis (i.e. along the z-direction) but logarithmically with respect to the extent of the side perpendicular to the axis (i.e. along the x-axis).

Example 26

If $\underline{F} = x^2\underline{i} + y^2\underline{j} + z^2\underline{k}$, evaluate $\displaystyle\iint_S \underline{F} \times \underline{dS}$ where S is the part of the plane $z = 0$ bounded by $x = \pm 1$, $y = \pm 1$.

Solution

Here $\underline{dS} = dx\,dy\,\underline{k}$ and hence $\quad \underline{F} \times \underline{dS} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x^2 & y^2 & z^2 \\ 0 & 0 & dx\,dy \end{vmatrix} = y^2\,dx\,dy\,\underline{i} - x^2\,dx\,dy\,\underline{j}$

$$\iint_S \underline{F} \times \underline{dS} = \int_{y=-1}^{1}\int_{x=-1}^{1} y^2\,dx\,dy\,\underline{i} - \int_{y=-1}^{1}\int_{x=-1}^{1} x^2\,dx\,dy\,\underline{j}$$

The first integral is

$$\int_{y=-1}^{1}\int_{x=-1}^{1} y^2\,dx\,dy = \int_{y=-1}^{1}\Big[y^2x\Big]_{x=-1}^{1} dy = \int_{y=-1}^{1} 2y^2 dy = \left[\frac{2}{3}y^3\right]_{-1}^{1} = \frac{4}{3}$$

Similarly $\displaystyle\int_{y=-1}^{1}\int_{x=-1}^{1} x^2\,dx\,dy = \frac{4}{3}$.

Thus $\displaystyle\iint_S \underline{F} \times \underline{dS} = \frac{4}{3}\underline{i} - \frac{4}{3}\underline{j}$

Key Point 5

(a) An integral of the form $\displaystyle\int_S \underline{F}(\underline{r}) \cdot \underline{dS}$ evaluates to a scalar.

(b) An integral of the form $\displaystyle\int_S \underline{F}(\underline{r}) \times \underline{dS}$ evaluates to a vector.

The vector function involved may be the gradient of a scalar or the curl of a vector.

Example 27

Integrate $\int\int_S (\underline{\nabla}\phi).\underline{dS}$ where $\phi = x^2 + 2yz$ and S is the area between $y = 0$ and $y = x^2$ for $0 \leq x \leq 1$ and $z = 0$. (See Figure 11.)

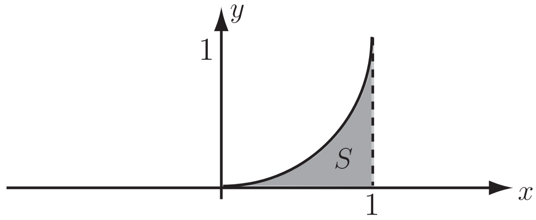

Figure 11: The area S between $y = 0$ and $y = x^2$, for $0 \leq x \leq 1$ and $z = 0$

Solution

Here $\underline{\nabla}\phi = 2x\underline{i} + 2z\underline{j} + 2y\underline{k}$ and $\underline{dS} = \underline{k}\ dydx$. Thus $(\underline{\nabla}\phi).\underline{dS} = 2ydydx$ and

$$\begin{aligned}\int\int_S (\underline{\nabla}\phi).\underline{dS} &= \int_{x=0}^{1}\int_{y=0}^{x^2} 2y\ dydx \\ &= \int_{x=0}^{1}\left[y^2\right]_{y=0}^{x^2} dx = \int_{x=0}^{1} x^4 dx \\ &= \left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5}\end{aligned}$$

For integrals of the form $\int\int_S \underline{F}\cdot\underline{dS}$, non-Cartesian coordinates e.g. cylindrical polar or spherical polar coordinates may be used. Once again, it is necessary to include any scale factors along with the unit normal.

Example 28

Using cylindrical polar coordinates, (see HELM 28.3), find the integral $\int_S \underline{F}(\underline{r})\cdot\underline{dS}$ for $\underline{F} = \rho z\underline{\hat{\rho}} + z\sin^2\phi\underline{\hat{z}}$ and S being the complete surface (including ends) of the cylinder $\rho \leq a$, $0 \leq z \leq 1$. (See Figure 12.)

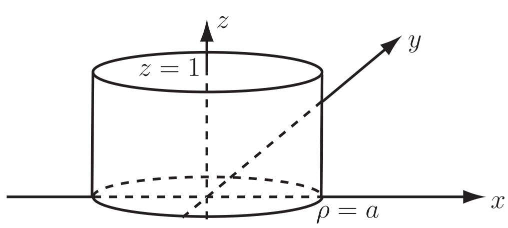

Figure 12: The cylinder $\rho \leq a,\ 0 \leq z \leq 1$

Solution

The integral $\int_S \underline{F}(\underline{r}) \cdot \underline{dS}$ must be evaluated separately for the curved surface and the ends.
For the curved surface, $\underline{dS} = \underline{\hat{\rho}} a d\phi dz$ (with the a coming from ρ the scale factor for ϕ and the fact that $\rho = a$ on the curved surface.) Thus, $\underline{F} \cdot \underline{dS} = a^2 z\ d\phi dz$ and

$$\begin{aligned}\iint_S \underline{F}(\underline{r}) \cdot \underline{dS} &= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} a^2 z\ d\phi dz \\ &= 2\pi a^2 \int_{z=0}^{1} z\ dz = 2\pi a^2 \left[\frac{1}{2} z^2\right]_0^1 = \pi a^2\end{aligned}$$

On the bottom surface, $z = 0$ so $\underline{F} = \underline{0}$ and the contribution to the integral is zero.
On the top surface, $z = 1$ and $\underline{dS} = \underline{\hat{z}} \rho\ d\rho d\phi$ and $\underline{F} \cdot \underline{dS} = \rho z \sin^2\phi\ d\phi d\rho = \rho \sin^2\phi\ d\phi d\rho$ and

$$\begin{aligned}\iint_S \underline{F}(\underline{r}) \cdot \underline{dS} &= \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \rho \sin^2\phi\ d\phi d\rho \\ &= \pi \int_{\rho=0}^{a} \rho\ d\rho = \frac{1}{2}\pi a^2\end{aligned}$$

So $$\iint_S \underline{F}(\underline{r}) \cdot \underline{dS} = \pi a^2 + \frac{1}{2}\pi a^2 = \frac{3}{2}\pi a^2$$

Engineering Example 4

The current continuity equation

Introduction

When an electric current flows at a constant rate through a conductor, then the **current continuity equation** states that

$$\oint_S \underline{J} \cdot \underline{dS} = 0$$

where $\underline{J}$ is the current density (or current flow per unit area) and S is a closed surface. The equation is an expression of the fact that, under these conditions, the current flow into a closed volume equals the flow out.

Problem in words

A person is standing nearby when lightning strikes the ground. Find the potential difference between the feet of that person.

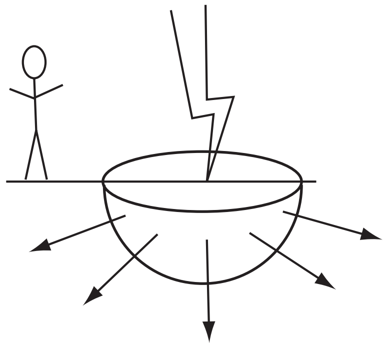

Figure 13: Lightning: a current dissipating into the ground

Mathematical statement of problem

The current from the lightning dissipates radially (see Fig 13).

(a) Find a relationship between the current I and current density J at a distance r from the strike by integrating the current density over the hemisphere $I = \int_S \underline{J} \cdot \underline{dS}$

(b) Find the field $\underline{E}$ from the equation $E = \dfrac{\rho I}{2\pi^2 r}$ where $E = |\underline{E}|$ and I is the current.

(c) Find V from the integral $\displaystyle\int_{R_1}^{R_2} \underline{E} \cdot \underline{dr}$

Mathematical analysis

Imagine a hemisphere of radius r level with the surface of the ground so that the point of lightning strike is at its centre. By symmetry, the pattern of current flow from the point of strike will be uniform radial lines, and the magnitude of $\underline{J}$ will be a constant, i.e. over the curved surface of the hemisphere $\underline{J} = J\underline{\hat{r}}$.

Since the amount of current entering the hemisphere is I, then it follows that the current leaving must be the same i.e.

$$\begin{aligned} I &= \int_{S_c} \underline{J} \cdot \underline{dS} \qquad \text{(where } S_c \text{ is the curved surface of the hemisphere)} \\ &= \int_{S_c} (J\underline{\hat{r}}) \cdot (dS\ \underline{\hat{r}}) \\ &= J \int_{S_c} dS \\ &= 2\pi r^2 J \qquad [= \text{surface area } (2\pi r^2) \times \text{flux } (J)] \end{aligned}$$

since the surface area of a sphere is $4\pi r^2$. Therefore

$$J = \frac{I}{2\pi r^2}$$

Note that if the current density $\underline{J}$ is uniformly radial over the curved surface, then so must be the electric field $\underline{E}$, i.e. $\underline{E} = E\underline{\hat{r}}$. Using Ohm's law

$$\underline{J} = \sigma \underline{E} \quad \text{or} \quad E = \rho J$$

where σ = conductivity = $1/\rho$.

Hence $\quad E = \dfrac{\rho I}{2\pi r^2}$

The potential difference between two points at radii R_1 and R_2 from the lightning strike is found by integrating $\underline{E}$ between them, so that

$$
\begin{aligned}
V &= \int_{R_1}^{R_2} \underline{E}\cdot \underline{dr} \\
&= \int_{R_1}^{R_2} E\, dr \\
&= \frac{\rho I}{2\pi}\int_{R_1}^{R_2} \frac{dr}{r^2} \\
&= \frac{\rho I}{2\pi}\left[\frac{-1}{r}\right]_{R_1}^{R_2} \\
&= \frac{\rho I}{2\pi}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \qquad \left(= \frac{\rho I}{2\pi}\left(\frac{R_2-R_1}{R_1R_2}\right)\right)
\end{aligned}
$$

Interpretation

Suppose the lightning strength is a current $I = 10,000$ A, that the person is 12 m away with feet 0.35 m apart, and that the resistivity of the ground is 80 Ω m. Clearly, the worst case (i.e. maximum voltage) would occur when the difference between R_1 and R_2 is greatest, i.e. R_1=12 m and R_2=12.35 m which would be the case if both feet were on the same radial line. The voltage produced between the person's feet under these circumstances is

$$
\begin{aligned}
V &= \frac{\rho I}{2\pi}\left[\frac{1}{R_1}-\frac{1}{R_2}\right] \\
&= \frac{80\times 10000}{2\pi}\left[\frac{1}{12}-\frac{1}{12.35}\right] \\
&\approx 300 \text{ V}
\end{aligned}
$$

Task

For $\underline{F} = (x^2+y^2)\underline{i} + (x^2+z^2)\underline{j} + 2xz\underline{k}$ and S the square bounded by $(1,0,1)$, $(1,0,-1)$, $(-1,0,-1)$ and $(-1,0,1)$ find the integral $\displaystyle\int_S \underline{F}\cdot\underline{dS}$

Your solution

Answer

$\underline{dS} = dxdz\underline{j} \qquad \displaystyle\int_{-1}^{1}\int_{-1}^{1}(x^2+z^2)\,dxdz = \frac{8}{3}$

Task

For $\underline{F} = (x^2+y^2)\underline{i} + (x^2+z^2)\underline{j} + 2xz\underline{k}$ and S being the rectangle bounded by $(1,0,1)$, $(1,0,-1)$, $(-1,0,-1)$ and $(-1,0,1)$ (i.e. the same $\underline{F}$ and S as in the previous Task), find the integral $\displaystyle\int_S \underline{F} \times \underline{dS}$

Your solution

Answer

$$\left\{\int_{-1}^{1}\int_{-1}^{1}(-2xz)\underline{i} + \int_{-1}^{1}\int_{-1}^{1}(x^2+0)\underline{k}\right\} dxdz = \frac{4}{3}\underline{k}$$

Exercises

1. Evaluate the integral $\displaystyle\iint_S \underline{\nabla}\phi \cdot \underline{dS}$ for $\phi = x^2 z \sin y$ and S being the rectangle bounded by $(0,0,0)$, $(1,0,1)$, $(1,\pi,1)$ and $(0,\pi,0)$.
2. Evaluate the integral $\displaystyle\iint_S (\underline{\nabla} \times \underline{F}) \times \underline{dS}$ where $\underline{F} = xe^y\underline{i} + ze^y\underline{j}$ and S represents the unit square $0 \le x \le 1$, $0 \le y \le 1$.
3. Using spherical polar coordinates (r, θ, ϕ), evaluate the integral $\displaystyle\iint_S \underline{F}\cdot\underline{dS}$ where $\underline{F} = r\cos\theta\underline{\hat{r}}$ and S is the curved surface of the top half of the sphere $r = a$.

Answers 1. $-\dfrac{2}{3}$, 2. $(e-1)\underline{j}$, 3. πa^3

2. Volume integrals involving vectors

Integrating a scalar function of a vector over a volume involves essentially the same procedure as in HELM 27.3. In 3D cartesian coordinates the volume element dV is $dxdydz$. The scalar function may be the divergence of a vector function.

Example 29

Integrate $\underline{\nabla} \cdot \underline{F}$ over the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ where $\underline{F}$ is the vector function $x^2y\underline{i} + (x-z)\underline{j} + 2xz^2\underline{k}$.

Solution

$$\underline{\nabla} \cdot \underline{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(x-z) + \frac{\partial}{\partial z}(2xz^2) = 2xy + 4xz$$

The integral is

$$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(2xy+4xz)dzdydx = \int_{x=0}^{1}\int_{y=0}^{1}\left[2xyz+2xz^2\right]_0^1 dydx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}(2xy+2x)\,dydx = \int_{x=0}^{1}\left[xy^2+2xy\right]_0^1 dx$$

$$= \int_{x=0}^{1} 3xdx = \left[\frac{3}{2}x^2\right]_0^1 = \frac{3}{2}$$

Key Point 6

The volume integral of a scalar function (including the divergence of a vector) is a scalar.

Task

Using spherical polar coordinates (r,θ,ϕ) and the vector field $\underline{F} = r^2\,\hat{\underline{r}} + r^2\sin\theta\,\hat{\underline{\theta}}$, evaluate the integral $\displaystyle\iiint_V \underline{\nabla} \cdot \underline{F}\,dV$ over the sphere given by $0 \le r \le a$.

Your solution

Answer

$\underline{\nabla} \cdot \underline{F} = 4r + 2r\cos\theta$, $\quad \displaystyle\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\{(4r+2r\cos\theta)r^2\sin\theta\}\,d\phi d\theta dr = 4\pi a^4$

The $r^2\sin\theta$ term comes from the Jacobian for the transformation from spherical to cartesian coordinates (see HELM 27.4 and HELM 28.3).

Exercises

1. Evaluate $\iiint_V \underline{\nabla} \cdot \underline{F} dV$ when $\underline{F}$ is the vector field $yz\underline{i} + xy\underline{j}$ and V is the unit cube $0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1$

2. For the vector field $\underline{F} = (x^2y + \sin z)\underline{i} + (xy^2 + e^z)\underline{j} + (z^2 + x^y)\underline{k}$, find the integral $\iiint_V \underline{\nabla} \cdot \underline{F} dV$ where V is the volume inside the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Answers 1. $\underline{\nabla} \cdot \underline{F} = x,\ \dfrac{1}{2}$ 2. $\dfrac{7}{60}$

Integrating a vector function over a volume integral is similar, but less common. Care should be taken with the various components. It may help to think in terms of a separate volume integral for each component. The vector function may be of the form $\underline{\nabla} f$ or $\underline{\nabla} \times \underline{F}$.

Example 30

Integrate the function $\underline{F} = x^2\underline{i} + 2\underline{j}$ over the prism given by $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq (1 - x)$. (See Figure 14.)

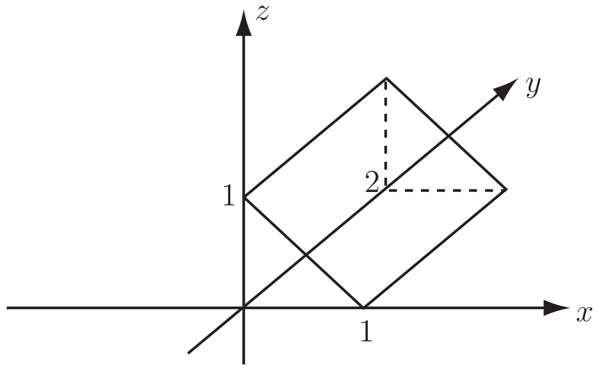

Figure 14: The prism bounded by $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq (1 - x)$

Solution

The integral is

$$\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{1-x}(x^2\underline{i} + 2\underline{j})dzdydx = \int_{x=0}^{1}\int_{y=0}^{2}\left[x^2z\underline{i} + 2z\underline{j}\right]_{z=0}^{1-x}dydx$$

$$= \int_{x=0}^{1}\int_{y=0}^{2}\left\{x^2(1-x)\underline{i} + 2(1-x)\underline{j}\right\}dydx = \int_{x=0}^{1}\int_{y=0}^{2}\left\{(x^2 - x^3)\underline{i} + (2 - 2x)\underline{j}\right\}dydx$$

$$= \int_{x=0}^{1}\left\{(2x^2 - 2x^3)\underline{i} + (4 - 4x)\underline{j}\right\}dx = \left[(\frac{2}{3}x^3 - \frac{1}{2}x^4)\underline{i} + (4x - 2x^2)\underline{j}\right]_0^1$$

$$= \frac{1}{6}\underline{i} + 2\underline{j}$$

Example 31

For $\underline{F} = x^2 y\underline{i} + y^2\underline{j}$ evaluate $\displaystyle\iiint_V (\underline{\nabla}\times\underline{F})dV$ where V is the volume under the plane $z = x + y + 2$ (and above $z = 0$) for $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

Solution

$$\underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & y^2 & 0 \end{vmatrix} = -x^2\underline{k}$$

so

$$\begin{aligned}
\iiint_V (\underline{\nabla}\times\underline{F})dV &= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=0}^{x+y+2} (-x^2)\underline{k}\; dzdydx \\
&= \int_{x=-1}^{1}\int_{y=-1}^{1} \Big[(-x^2)z\underline{k}\Big]_{z=0}^{x+y+2} dydx \\
&= \int_{x=-1}^{1}\int_{y=-1}^{1} \left[-x^3 - x^2y - 2x^2\right] dydx\;\underline{k} \\
&= \int_{x=-1}^{1} \left[-x^3y - \frac{1}{2}x^2y^2 - 2x^2y\right]_{y=-1}^{1} dx\;\underline{k} \\
&= \int_{x=-1}^{1} \left(-2x^3 - 0 - 4x^2\right) dx\;\underline{k} = \left[-\frac{1}{2}x^4 - \frac{4}{3}x^3\right]_{-1}^{1}\underline{k} = -\frac{8}{3}\underline{k}
\end{aligned}$$

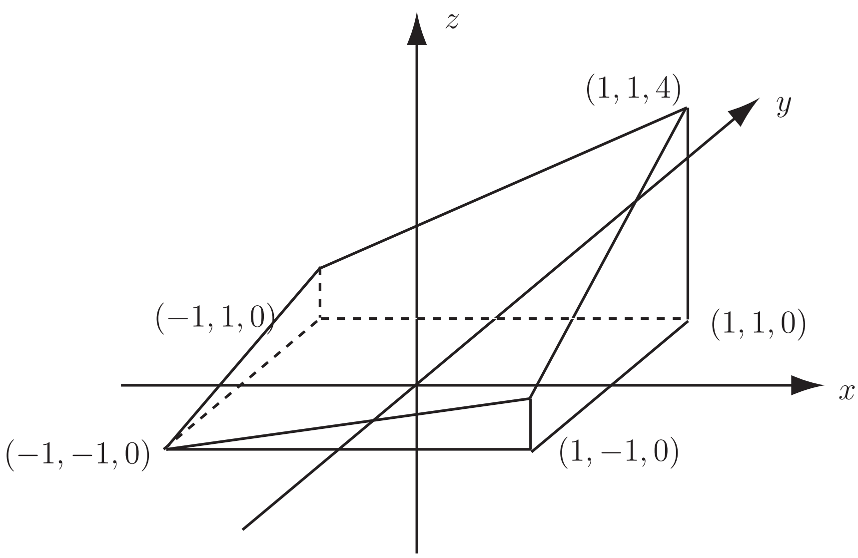

Figure 15: The plane defined by $z = x + y + z$, for $z > 0$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$

Key Point 7

The volume integral of a vector function (including the gradient of a scalar or the curl of a vector) is a vector.

Task

Evaluate the integral $\int_V \underline{F} dV$ for the case where $\underline{F} = x\underline{i} + y^2\underline{j} + z\underline{k}$ and V is the cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

Your solution

Answer

$$\int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} (x\underline{i} + y^2\underline{j} + z\underline{k}) dz dy dx = \frac{8}{3}\underline{j}$$

Exercises

1. For $f = x^2 + yz$, and V the volume bounded by $y = 0$, $x + y = 1$ and $-x + y = 1$ for $-1 \leq z \leq 1$, find the integral $\iiint_V (\underline{\nabla} f) dV$.

2. Evaluate the integral $\int_V (\underline{\nabla} \times \underline{F}) dV$ for the case where $\underline{F} = xz\underline{i} + (x^3 + y^3)\underline{j} - 4y\underline{k}$ and V is the cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

Answers

1. $\iiint_V (2x\underline{i} + z\underline{j} + y\underline{k}) dV = \frac{2}{3}\underline{k}$,

2. $\iiint_V (-4\underline{i} + x\underline{j} + 3x^2\underline{k}) dV = -32\underline{i} + 8\underline{k}$

Integral Vector Theorems

29.3

Introduction

Various theorems exist relating integrals involving vectors. Those involving line, surface and volume integrals are introduced here.

They are the multivariable calculus equivalent of the fundamental theorem of calculus for single variables ("integration and differentiation are the reverse of each other").

Use of these theorems can often make evaluation of certain vector integrals easier. This Section introduces the main theorems which are Gauss' divergence theorem, Stokes' theorem and Green's theorem.

Prerequisites

Before starting this Section you should ...

- be able to find the gradient of a scalar field and the divergence and curl of a vector field
- be familiar with the integration of vector functions

Learning Outcomes

On completion you should be able to ...

- use vector integral theorems to facilitate vector integration

1. Stokes' theorem

This is a theorem that equates a line integral to a surface integral. For any vector field $\underline{F}$ and a contour C which bounds an area S,

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

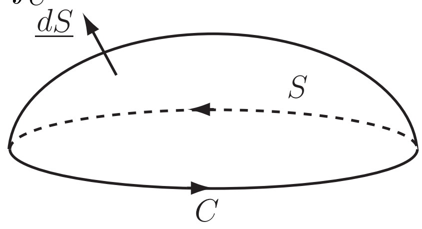

Figure 16: A surface for Stokes' theorem

Notes

(a) $\underline{dS}$ is a vector perpendicular to the surface S and $\underline{dr}$ is a line element along the contour C. The sense of $\underline{dS}$ is linked to the direction of travel along C by a right hand screw rule.

(b) Both sides of the equation are scalars.

(c) The theorem is often a useful way of calculating a line integral along a contour composed of several distinct parts (e.g. a square or other figure).

(d) $\underline{\nabla} \times \underline{F}$ is a vector field representing the curl of the vector field $\underline{F}$ and may, alternatively, be written as curl $\underline{F}$.

Justification of Stokes' theorem

Imagine that the surface S is divided into a set of infinitesimally small rectangles $ABCD$ where the axes are adjusted so that AB and CD lie parallel to the new x-axis i.e. $AB = \delta x$ and BC and AD lie parallel to the new y-axis i.e. $BC = \delta y$.

Now, $\displaystyle\oint_C \underline{F} \cdot \underline{dr}$ is calculated, where C is the boundary of a typical such rectangle.

The contributions along AB, BC, CD and DA are

$$\underline{F}(x, y, 0) \cdot \underline{\delta x} = F_x(x, y, z)\delta x,$$

$$\underline{F}(x + \delta x, y, 0) \cdot \underline{\delta y} = F_y(x + \delta x, y, z)\delta y,$$

$$\underline{F}(x, y + \delta y, 0) \cdot (-\underline{\delta x}) = -F_x(x, y + \delta y, z)\delta x$$

$$\underline{F}(x, y, 0) \cdot (-\underline{\delta x}) = -F_y(x, y, z)\delta y.$$

Thus,

$$\begin{aligned}
\oint_C \underline{F} \cdot \underline{dr} &\approx (F_x(x, y, z) - F_x(x, y + \delta y, z))\delta x + (F_y(x + \delta x, y, z) - F_y(x, y, z))\delta y \\
&\approx \frac{\partial F_y}{\partial x}\delta x\delta y - \frac{\partial F_x}{\partial y}\delta x\delta y \\
&\approx (\underline{\nabla} \times \underline{F})_z \delta S \\
&= (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}
\end{aligned}$$

as $\underline{dS}$ is perpendicular to the x- and y- axes.

Thus, for each small rectangle, $\oint_C \underline{F} \cdot \underline{dr} \approx (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$

When the contributions over all the small rectangles are summed, the line integrals along the inner parts of the rectangles cancel and all that remains is the line integral around the outside of the surface S. The surface integrals sum. Hence, the theorem applies for the area S bounded by the contour C. While the above does not constitute a formal proof of Stokes' theorem, it does give an appreciation of the origin of the theorem.

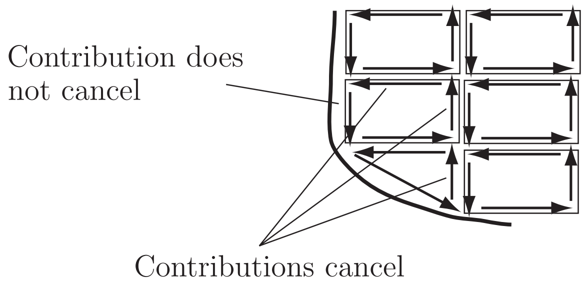

Figure 17: Line integral cancellation and non-cancellation

Key Point 8

Stokes' Theorem

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$$

The closed contour integral of the scalar product of a vector function with the vector along the contour is equal to the integral of the scalar product of the curl of that vector function and the unit normal, over the corresponding surface.

Example 32

Verify Stokes' theorem for the vector function $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$ and the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$.

Solution

If $\underline{F} = y^2\underline{i} - (x+z)\underline{j} + yz\underline{k}$ then $\underline{\nabla} \times \underline{F} = (z+1)\underline{i} + (-1-2y)\underline{k} = \underline{i} + (-1-2y)\underline{k}$ (as $z = 0$).
Note that $\underline{dS} = dxdy\underline{k}$ so that $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = (-1-2y)dydx$

Thus
$$\begin{aligned}\int\!\!\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} &= \int_{x=0}^{1}\int_{y=0}^{1}(-1-2y)dydx \\ &= \int_{x=0}^{1}\Big[(-y-y^2)\Big]_{y=0}^{1} dx = \int_{x=0}^{1}(-2)dx \\ &= \Big[-2x\Big]_0^1 = -2+0 = -2\end{aligned}$$

To evaluate $\oint_C \underline{F} \cdot \underline{dr}$, we must consider the four sides separately.
When $y = 0$, $\underline{F} = -x\underline{j}$ and $\underline{dr} = dx\underline{i}$ so $\underline{F} \cdot \underline{dr} = 0$ i.e. the contribution of this side to the integral is zero.
When $x = 1$, $\underline{F} = y^2\underline{i} - \underline{j}$ and $\underline{dr} = dy\underline{j}$ so $\underline{F} \cdot \underline{dr} = -dy$ so the contribution to the integral is

$$\int_{y=0}^{1}(-dy) = \Big[-y\Big]_0^1 = -1.$$

When $y = 1$, $\underline{F} = \underline{i} - x\underline{j}$ and $\underline{dr} = -dx\underline{i}$ so $\underline{F} \cdot \underline{dr} = -dx$ so the contribution to the integral is

$$\int_{x=0}^{1}(-dx) = \Big[-x\Big]_0^1 = -1.$$

When $x = 0$, $\underline{F} = y^2\underline{i}$ and $\underline{dr} = -dy\underline{j}$ so $\underline{F} \cdot \underline{dr} = 0$ so the contribution to the integral is zero.

The integral $\oint_C \underline{F} \cdot \underline{dr}$ is the sum of the contributions i.e. $0 - 1 - 1 + 0 = -2$.

Thus $\displaystyle\int\!\!\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr} = -2$ i.e. Stokes' theorem has been verified.

Example 33

Using cylindrical polar coordinates verify Stokes' theorem for the function $\underline{F} = \rho^2\underline{\hat{\phi}}$ the circle $\rho = a, z = 0$ and the surface $\rho \leq a, z = 0$.

Solution

Firstly, find $\oint_C \underline{F} \cdot \underline{dr}$. This can be done by integrating along the contour $\rho = a$ from $\phi = 0$ to $\phi = 2\pi$. Here $\underline{F} = a^2\underline{\hat{\phi}}$ (as $\rho = a$) and $\underline{dr} = a\ d\phi\ \underline{\hat{\phi}}$ (remembering the scale factor) so $\underline{F}\cdot\underline{dr} = a^3 d\phi$ and hence

$$\oint_C \underline{F} \cdot \underline{dr} = \int_0^{2\pi} a^3 d\phi = 2\pi a^3$$

As $\underline{F} = \rho^2\underline{\hat{\phi}}$, $\underline{\nabla} \times \underline{F} = 3\rho\underline{\hat{z}}$ and $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 3\rho$ as $\underline{dS} = \underline{\hat{z}}$.
Thus

$$\begin{aligned}\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} &= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{1} 3\rho \times \rho d\rho d\phi = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} 3\rho^2 d\rho d\phi \\ &= \int_{\phi=0}^{2\pi} \Big[\rho^3\Big]_{\rho=0}^{a} d\phi = \int_0^{2\pi} a^3 d\phi = 2\pi a^3\end{aligned}$$

Hence

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 2\pi a^3$$

Example 34

Find the closed line integral $\oint_C \underline{F}\cdot\underline{dr}$ for the vector field $\underline{F} = y^2\underline{i} + (x^2 - z)\underline{j} + 2xy\underline{k}$ and for the contour $ABCDEFGHA$ in Figure 18.

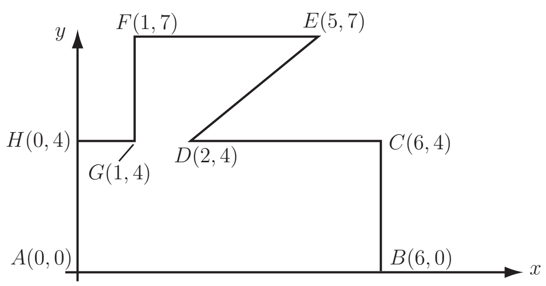

Figure 18: Closed contour $ABCDEFGHA$

Solution

To find the line integral directly would require eight line integrals i.e. along AB, BC, CD, DE, EF, FG, GH and HA. It is easier to carry out a surface integral to find $\int\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ which is equal to the required line integral $\oint_C \underline{F} \cdot \underline{dr}$ by Stokes' theorem.

As $\underline{F} = y^2\underline{i} + (x^2 - z)\underline{j} + 2xy\underline{k}$, $\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 - z & 2xy \end{vmatrix} = (2x+1)\underline{i} - 2y\underline{j} + (2x - 2y)\underline{k}$

As the contour lies in the x-y plane, the unit normal is $\underline{k}$ and $\underline{dS} = dxdy\underline{k}$

Hence $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = (2x - 2y)dxdy$.

To work out $\int\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$, it is necessary to divide the area inside the contour into two smaller areas i.e. the rectangle $ABCDGH$ and the trapezium $DEFG$. On $ABCDGH$, the integral is

$$\begin{aligned} \int_{y=0}^{4} \int_{x=0}^{6} (2x - 2y)dxdy &= \int_{y=0}^{4} \left[x^2 - 2xy \right]_{x=0}^{6} dy = \int_{y=0}^{4} (36 - 12y)dy \\ &= \left[36y - 6y^2 \right]_0^4 = 36 \times 4 - 6 \times 16 - 0 = 48 \end{aligned}$$

On $DEFG$, the integral is

$$\begin{aligned} \int_{y=4}^{7} \int_{x=1}^{y-2} (2x - 2y)dxdy &= \int_{y=4}^{7} \left[x^2 - 2xy \right]_{x=1}^{y-2} dy = \int_{y=4}^{7} (-y^2 + 2y + 3)dy \\ &= \left[-\frac{1}{3}y^3 + y^2 + 3y \right]_4^7 = -\frac{343}{3} + 49 + 21 + \frac{64}{3} - 16 - 12 = -51 \end{aligned}$$

So the full integral is, $\int\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = 48 - 51 = -3$.

$\therefore$ By Stokes' theorem, $\oint_C \underline{F} \cdot \underline{dr} = -3$

From Stokes' theorem, it can be seen that surface integrals of the form $\int\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ depend only on the contour bounding the surface and not on the internal part of the surface.

Task

Verify Stokes' theorem for the vector field $\underline{F} = x^2\underline{i} + 2xy\underline{j} + z\underline{k}$ and the triangle with vertices at $(0, 0, 0)$, $(3, 0, 0)$ and $(3, 1, 0)$.

First find the normal vector $\underline{dS}$:

Your solution

Answer

$dxdy\underline{k}$

Then find the vector $\underline{\nabla} \times \underline{F}$:

Your solution

Answer

$2y\underline{k}$

Now evaluate the double integral $\displaystyle\int\!\!\int_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS}$ over the triangle:

Your solution

Answer

1

Finally find the integral $\int \underline{F} \cdot \underline{dr}$ along the 3 sides of the triangle and so verify that the two sides of the Stokes' theorem are equal:

Your solution

Answer

$9 + 3 - 11 = 1,$ Both sides of Stokes' theorem have value 1.

Exercises

1. Using plane-polar coordinates (or cylindrical polar coordinates with $z = 0$), verify Stokes' theorem for the vector field $\underline{F} = \rho\hat{\underline{\rho}} + \rho\cos\left(\dfrac{\pi\rho}{2}\right)\hat{\underline{\phi}}$ and the semi-circle $\rho \leq 1$, $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$.
2. Verify Stokes' theorem for the vector field $\underline{F} = 2x\underline{i} + (y^2 - z)\underline{j} + xz\underline{k}$ and the contour around the rectangle with vertices at $(0, -2, 0)$,$(2, -2, 0)$, $(2, 0, 1)$ and $(0, 0, 1)$.
3. Verify Stokes' theorem for the vector field $\underline{F} = -y\underline{i} + x\underline{j} + z\underline{k}$
   (a) Over the triangle $(0, 0, 0), (1, 0, 0), (1, 1, 0)$.
   (b) Over the triangle $(1, 0, 0), (1, 1, 0), (1, 1, 1)$.
4. Use Stokes' theorem to evaluate the integral

   $$\oint_C \underline{F} \cdot \underline{dr} \quad \text{where} \quad \underline{F} = \left(\sin(\frac{1}{x} + 1) + 5y\right)\underline{i} + (2x - e^{y^2})\underline{j}$$

   and C is the contour starting at $(0, 0)$ and going to $(5, 0)$, $(5, 2)$, $(6, 2)$, $(6, 5)$, $(3, 5)$, $(3, 2)$, $(0, 2)$ and returning to $(0, 0)$.

Answers

1. Both integrals give 0,

2. Both integrals give 1

3. (a) Both integrals give 1 (b) Both integrals give 0 (as $\underline{\nabla} \times \underline{F}$ is perpendicular to $\underline{dS}$)

4. -57, $[\underline{\nabla} \times \underline{F} = -3\,\underline{k}]$.

2. Gauss' theorem

This is sometimes known as the **divergence theorem** and is similar in form to Stokes' theorem but equates a surface integral to a volume integral. Gauss' theorem states that for a volume V, bounded by a closed surface S, any 'well-behaved' vector field $\underline{F}$ satisfies

$$\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F}\, dV$$

Notes:

(a) $\underline{dS}$ is a unit normal pointing outwards from the interior of the volume V.

(b) Both sides of the equation are scalars.

(c) The theorem is often a useful way of calculating a surface integral over a surface composed of several distinct parts (e.g. a cube).

(d) $\underline{\nabla} \cdot \underline{F}$ is a scalar field representing the divergence of the vector field $\underline{F}$ and may, alternatively, be written as div $\underline{F}$.

(e) Gauss' theorem can be justified in a manner similar to that used for Stokes' theorem (i.e. by proving it for a small volume element, then summing up the volume elements and allowing the internal surface contributions to cancel.)

Key Point 9

Gauss' Theorem

$$\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F} dV$$

The closed surface integral of the scalar product of a vector function with the unit normal (or flux of a vector function through a surface) is equal to the integral of the divergence of that vector function over the corresponding volume.

Example 35

Verify Gauss' theorem for the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$ and the function $\underline{F} = x\underline{i} + z\underline{j}$

Solution

To find $\displaystyle\iint_S \underline{F} \cdot \underline{dS}$, the integral must be evaluated for all six faces of the cube and the results summed.

On the face $x = 0$, $\underline{F} = z\underline{j}$ and $\underline{dS} = -\underline{i}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\; dydz = 0$$

On the face $x = 1$, $\underline{F} = \underline{i} + z\underline{j}$ and $\underline{dS} = \underline{i}\, dydz$ so $\underline{F} \cdot \underline{dS} = 1\; dydz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 1\; dydz = 1$$

On the face $y = 0$, $\underline{F} = x\underline{i} + z\underline{j}$ and $\underline{dS} = -\underline{j}\, dxdz$ so $\underline{F} \cdot \underline{dS} = -z\; dxdz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = -\int_0^1 \int_0^1 z\; dxdz = -\frac{1}{2}$$

On the face $y = 1$, $\underline{F} = x\underline{i} + z\underline{j}$ and $\underline{dS} = \underline{j}\, dxdz$ so $\underline{F} \cdot \underline{dS} = z\; dxdz$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 z\; dxdz = \frac{1}{2}$$

On the face $z = 0$, $\underline{F} = x\underline{i}$ and $\underline{dS} = -\underline{k}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0\; dxdy$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\; dxdy = 0$$

On the face $z = 1$, $\underline{F} = x\underline{i} + \underline{j}$ and $\underline{dS} = \underline{k}\, dydz$ so $\underline{F} \cdot \underline{dS} = 0\; dxdy$ and

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 0\; dxdy = 0$$

Thus, summing over all six faces, $\displaystyle\iint_S \underline{F} \cdot \underline{dS} = 0 + 1 - \frac{1}{2} + \frac{1}{2} + 0 + 0 = 1.$

To find $\displaystyle\iiint_V \underline{\nabla} \cdot \underline{F}\; dV$ note that $\underline{\nabla} \cdot \underline{F} = \dfrac{\partial}{\partial x}x + \dfrac{\partial}{\partial y}z = 1 + 0 = 1.$

So $\displaystyle\iiint_V \underline{\nabla} \cdot \underline{F}\; dV = \int_0^1 \int_0^1 \int_0^1 1\; dxdydz = 1.$

So $\displaystyle\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F}\; dV = 1$ hence verifying Gauss' theorem.

Note: The volume integral needed just one triple integral, but the surface integral required six double integrals. Reducing the number of integrals is often the motivation for using Gauss' theorem.

Example 36

Use Gauss' theorem to evaluate the surface integral $\int\int_S \underline{F} \cdot \underline{dS}$ where $\underline{F}$ is the vector field $x^2y\underline{i} + 2xy\underline{j} + z^3\underline{k}$ and S is the surface of the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

Note that to carry out the surface integral directly will involve, as in Example 35, the evaluation of six double integrals. However, by Gauss' theorem, the same result comes from the volume integral $\int\int\int_V \underline{\nabla} \cdot \underline{F}\, dV$. As $\underline{\nabla} \cdot \underline{F} = 2xy + 2x + 3z^2$, we have the triple integral

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_0^1 (2xy + 2x + 3z^2)\, dxdydz \\
= & \int_0^1 \int_0^1 \Big[x^2y + x^2 + 3xz^2 \Big]_{x=0}^{1} dydz = \int_0^1 \int_0^1 (y + 1 + 3z^2) dydz \\
= & \int_0^1 \left[\frac{1}{2}y^2 + y + 3yz^2 \right]_{y=0}^{1} dz = \int_0^1 (\frac{1}{2} + 1 + 3z^2) dz = \int_0^1 (\frac{3}{2} + 3z^2) dz \\
= & \left[\frac{3}{2}z + z^3 \right]_0^1 = \frac{5}{2}
\end{aligned}$$

The six double integrals would also sum to $\frac{5}{2}$ but this approach would require much more effort.

Engineering Example 5

Gauss' law

Introduction

From Gauss' theorem, it is possible to derive a result which can be used to gain insight into situations arising in Electrical Engineering. Knowing the electric field on a closed surface, it is possible to find the electric charge within this surface. Alternatively, in a sufficiently symmetrical situation, it is possible to find the electric field produced by a given charge distribution.
Gauss' theorem states

$$\int\int_S \underline{F} \cdot \underline{dS} = \int\int\int_V \underline{\nabla} \cdot \underline{F}\, dV$$

If $\underline{F} = \underline{E}$, the electric field, it can be shown that,

$$\underline{\nabla} \cdot \underline{F} = \underline{\nabla} \cdot \underline{E} = \frac{q}{\varepsilon_0}$$

where q is the amount of charge per unit volume, or charge density, and ε_0 is the permittivity of free space: $\varepsilon_0 = 10^{-9}/36\pi$ F m^{-1} $\approx$ 8.84$\times 10^{-12}$ F m^{-1}. Gauss' theorem becomes in this case

$$\int\int_S \underline{E} \cdot \underline{dS} = \int\int\int_V \underline{\nabla} \cdot \underline{E}\, dV = \int\int\int_V \frac{q}{\varepsilon_0}\, dV = \frac{1}{\varepsilon_0}\int\int\int_V q\, dV = \frac{Q}{\varepsilon_0}$$

i.e.

$$\int\int_S \underline{E} \cdot \underline{dS} = \frac{Q}{\varepsilon_0}$$

which is known as Gauss' law. Here Q is the total charge inside the surface S.

Note: this is one of the important Maxwell's Laws.

Problem in words

A point charge lies at the centre of a cube. Given the electric field, find the magnitude of the charge, using Gauss' law .

Mathematical statement of problem

Consider the cube $-\frac{1}{2} \le x \le \frac{1}{2}$, $-\frac{1}{2} \le y \le \frac{1}{2}$, $-\frac{1}{2} \le z \le \frac{1}{2}$ where the dimensions are in metres. A point charge Q lies at the centre of the cube. If the electric field on the top face $(z = \frac{1}{2})$ is given by

$$\underline{E} = 10\frac{x\underline{i} + y\underline{j} + z\underline{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

find the charge Q from Gauss' law .

$$\left[\text{Hint : } \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=-\frac{1}{2}}^{\frac{1}{2}} \left(x^2 + y^2 + \frac{1}{4}\right)^{-\frac{3}{2}}\, dy\, dx = \frac{4\pi}{3}\right]$$

Mathematical analysis

From Gauss' law

$$\int\int_S \underline{E} \cdot \underline{dS} = \frac{Q}{\varepsilon_0}$$

so

$$Q = \varepsilon_0 \int\int_S \underline{E} \cdot \underline{dS} = 6\varepsilon_0 \int\int_{S(top)} \underline{E} \cdot \underline{dS}$$

since, using the symmetry of the six faces of the cube, it is possible to integrate over just one of them (here the top face is chosen) and multiply by 6. On the top face

$$\underline{E} = 10\frac{x\underline{i} + y\underline{j} + \frac{1}{2}\underline{k}}{\left(x^2 + y^2 + \frac{1}{4}\right)^{\frac{3}{2}}}$$

and

$$\begin{aligned} \underline{dS} &= \text{(element of surface area)} \times \text{(unit normal)} \\ &= dx\, dy\, \underline{k} \end{aligned}$$

So

$$\begin{aligned}\underline{E}\cdot\underline{dS} &= 10\frac{\frac{1}{2}}{\left(x^2+y^2+\frac{1}{4}\right)^{\frac{3}{2}}}\,dy\,dx \\ &= 5\left(x^2+y^2+\frac{1}{4}\right)^{-\frac{3}{2}}\,dy\,dx\end{aligned}$$

Now

$$\begin{aligned}\int\int_{S(top)}\underline{E}\cdot\underline{dS} &= \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=-\frac{1}{2}}^{\frac{1}{2}} 5\left(x^2+y^2+\frac{1}{4}\right)^{-\frac{3}{2}}\,dy\,dx \\ &= 5\times\frac{4\pi}{3}\quad\text{(using the hint)} \\ &= \frac{20\pi}{3}\end{aligned}$$

So, from Gauss' law,

$$Q = 6\varepsilon_0\times\frac{20\pi}{3} = 40\pi\varepsilon_0 \approx 10^{-9}\text{C}$$

Interpretation

Gauss' law can be used to find a charge from its effects elsewhere.

The form of $\underline{E} = 10\dfrac{x\underline{i}+y\underline{j}+\frac{1}{2}\underline{k}}{\left(x^2+y^2+\frac{1}{4}\right)^{\frac{3}{2}}}$ comes from the fact that $\underline{E}$ is radial and equals $10\dfrac{\underline{r}}{r^3} = 10\dfrac{\hat{\underline{r}}}{r^2}$

Example 37

Verify Gauss' theorem for the vector field $\underline{F} = y^2\underline{j} - xz\underline{k}$ and the triangular prism with vertices at $(0,0,0)$, $(2,0,0)$, $(0,0,1)$, $(0,4,0)$, $(2,4,0)$ and $(0,4,1)$ (see Figure 19).

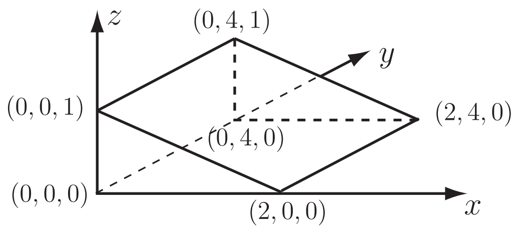

Figure 19: The triangular prism defined by six vertices

Solution

As $\underline{F} = y^2\underline{j} - xz\underline{k}$, $\underline{\nabla} \cdot \underline{F} = 0 + 2y - x = 2y - x$.
Thus

$$\begin{aligned}
\iiint_V \underline{\nabla} \cdot \underline{F} dV &= \int_{x=0}^{2}\int_{y=0}^{4}\int_{z=0}^{1-x/2}(2y-x)dzdydx \\
&= \int_{x=0}^{2}\int_{y=0}^{4}\Big[2yz - xz\Big]_{z=0}^{1-x/2} dydx = \int_{x=0}^{2}\int_{y=0}^{4}(2y - xy - x + \frac{1}{2}x^2)dydx \\
&= \int_{x=0}^{2}\left[y^2 - \frac{1}{2}xy^2 - xy + \frac{1}{2}x^2y\right]_{y=0}^{4} dx \quad = \int_{x=0}^{2}(16 - 12x + 2x^2)dx \\
&= \left[16x - 6x^2 + \frac{2}{3}x^3\right]_0^2 = \frac{40}{3}
\end{aligned}$$

To work out $\iint_S \underline{F} \cdot \underline{dS}$, it is necessary to consider the contributions from the five faces separately.
On the front face, $y = 0$, $\underline{F} = -xz\underline{k}$ and $\underline{dS} = -\underline{j}$ thus $\underline{F} \cdot \underline{dS} = 0$ and the contribution to the integral is zero.
On the back face, $y = 4$, $\underline{F} = 16\underline{j} - xz\underline{k}$ and $\underline{dS} = \underline{j}$ thus $\underline{F} \cdot \underline{dS} = 16$ and the contribution to the integral is

$$\int_{x=0}^{2}\int_{z=0}^{1-x/2} 16dzdx = \int_{x=0}^{2}\Big[\ 16z\Big]_{z=0}^{1-x/2} dx = \int_{x=0}^{2}16(1-x/2)dx = \Big[\ 16x - 4x^2\Big]_0^2 = 16.$$

On the left face, $x = 0$, $\underline{F} = y^2\underline{j}$ and $\underline{dS} = -\underline{i}$ thus $\underline{F} \cdot \underline{dS} = 0$ and the contribution to the integral is zero.
On the bottom face, $z = 0$, $\underline{F} = y^2\underline{j}$ and $\underline{dS} = -\underline{k}$ thus $\underline{F} \cdot \underline{dS} = 0$ and the contribution to the integral is zero.
On the top right (sloping) face, $z = 1 - x/2$, $\underline{F} = y^2\underline{j} + (\frac{1}{2}x^2 - x)\underline{k}$ and the unit normal $\underline{\hat{n}} = \frac{1}{\sqrt{5}}\underline{i} + \frac{2}{\sqrt{5}}\underline{k}$
Thus $\underline{dS} = \left[\frac{1}{\sqrt{5}}\underline{i} + \frac{2}{\sqrt{5}}\underline{k}\right] dydw$ where dw measures the distance along the slope for a constant y.
As $dw = \frac{\sqrt{5}}{2}dx$, $\underline{dS} = \left[\frac{1}{2}\underline{i} + \underline{k}\right] dydx$ thus $\underline{F} \cdot \underline{dS} = 16$ and the contribution to the integral is

$$\int_{x=0}^{2}\int_{y=0}^{4}(\frac{1}{2}x^2 - x)dydx = \int_{x=0}^{2}(2x^2 - 4x)dx = \left[\frac{2}{3}x^3 - 2x^2\right]_0^2 = -\frac{8}{3}.$$

Adding the contributions, $\iint_S \underline{F} \cdot \underline{dS} = 0 + 16 + 0 + 0 - \dfrac{8}{3} = \dfrac{40}{3}$.

Thus $\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F} dV = \dfrac{40}{3}$ hence verifying Gauss' divergence theorem.

Engineering Example 6

Field strength around a charged line

Problem in words

Find the electric field strength at a given distance from a uniformly charged line.

Mathematical statement of problem

Determine the electric field at a distance r from a uniformly charged line (charge per unit length ρ_L). You may assume from symmetry that the field points directly away from the line.

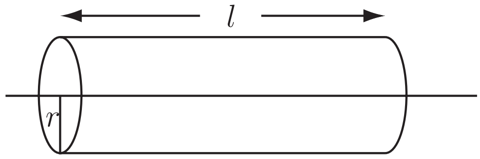

Figure 20: Field strength around a line charge

Mathematical analysis

Imagine a cylinder a distance r from the line and of length l (see Figure 20). From Gauss' law

$$\int\int_S \underline{E}\cdot\underline{dS} = \frac{Q}{\varepsilon_0}$$

As the charge per unit length is ρ_L, then the right-hand side equals $\rho_L l/\varepsilon_0$. On the left-hand side, the integral can be expressed as the sum

$$\int\int_S \underline{E}\cdot\underline{dS} = \int\int_{S(ends)} \underline{E}\cdot\underline{dS} \;+\; \int\int_{S(curved)} \underline{E}\cdot\underline{dS}$$

Looking first at the circular ends of the cylinder, the fact that the field lines point radially away from the charged line implies that the electric field is in the plane of these circles and has no normal component. Therefore $\underline{E}\cdot\underline{dS}$ will be zero for these ends.
Next, over the curved surface of the cylinder, the electric field is normal to it, and the symmetry of the problem implies that the strength of the electric field will be constant (here denoted by E). Therefore the integral = Total curved surface area × Field strength = $2\pi rlE$.

So, by Gauss' law

$$\int\int_{S(ends)} \underline{E}\cdot\underline{dS} \;+\; \int\int_{S(curved)} \underline{E}\cdot\underline{dS} = \frac{Q}{\varepsilon_0}$$

or

$$0 \;+\; 2\pi rlE = \frac{\rho_L l}{\varepsilon_0}$$

Interpretation

Hence, the field strength E is given by $\quad E = \dfrac{\rho_L}{2\pi\varepsilon_0 r}$

Engineering Example 7

Field strength on a cylinder

Problem in words

Given the electric field $\underline{E}$ on the surface of a cylinder, use Gauss' law to find the charge per unit length.

Mathematical statement of problem

On the surface of a long cylinder of radius a and length l, the electric field is given by

$$\underline{E} = \frac{\rho_L}{2\pi\varepsilon_0}\,\frac{(a + b\cos\theta)\,\hat{\underline{r}} - b\sin\theta\,\hat{\underline{\theta}}}{(a^2 + 2ab\cos\theta + b^2)}$$

(using cylindrical polar co-ordinates) due to a line of charge a distance b ($< a$) from the centre of the cylinder. Using Gauss' law , find the charge per unit length.

Hint:- $$\int_0^{2\pi} \frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\,d\theta = \frac{2\pi}{a}$$

Mathematical analysis

Consider a cylindrical section - as in the previous example, there are no contributions from the ends of the cylinder since the electric field has no normal component here. However, on the curved surface

$$\underline{dS} = a\;d\theta\;dz\;\hat{\underline{r}}$$

so

$$\underline{E}\cdot\underline{dS} = \frac{\rho_L}{2\pi\varepsilon_0}\,\frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\;a\;d\theta\;dz$$

Integrating over the curved surface of the cylinder

$$\begin{aligned}
\int\!\!\int_S \underline{E}\cdot\underline{dS} &= \int_{z=0}^{l}\int_{\theta=0}^{\theta=2\pi} \frac{a\rho_L}{2\pi\varepsilon_0}\,\frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\;d\theta\;dz \\
&= \frac{a\rho_L l}{2\pi\varepsilon_0}\int_0^{2\pi} \frac{a + b\cos\theta}{(a^2 + 2ab\cos\theta + b^2)}\;d\theta \\
&= \frac{\rho_L l}{\varepsilon_0} \quad \text{using the given result for the integral.}
\end{aligned}$$

Then, if Q is the total charge inside the cylinder, from Gauss' law

$$\frac{\rho_L l}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \quad \text{so} \quad \rho_L = \frac{Q}{l} \quad \text{as one would expect.}$$

Interpretation

Therefore the charge per unit length on the line of charge is given by ρ_L (i.e. the charge per unit length is constant).

Task

Verify Gauss' theorem for the vector field $\underline{F} = x\underline{i} - y\underline{j} + z\underline{k}$ and the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

(a) Find the vector $\underline{\nabla} \cdot \underline{F}$.

(b) Evaluate the integral $\displaystyle\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} \underline{\nabla} \cdot \underline{F} dxdydz$.

(c) For each side, evaluate the normal vector $\underline{dS}$ and the surface integral $\displaystyle\iint_S \underline{F} \cdot \underline{dS}$.

(d) Show that the two sides of the statement of Gauss' theorem are equal.

Your solution

Answer

(a) $1 - 1 + 1 = 1$

(b) 1

(c) $-dxdy\underline{k},\ 0;\quad dxdy\underline{k},\quad 1;\quad -dxdz\underline{j},\ 0;\quad dxdz\underline{j},\quad -1;\quad -dydz\underline{i},\ 0;\quad dydz\underline{i},\quad 1$

(d) Both sides are equal to 1.

Exercises

1. Verify Gauss' theorem for the vector field $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and the cuboid $0 \leq x \leq 2$, $0 \leq y \leq 3$, $0 \leq z \leq 4$.

2. Verify Gauss' theorem, using cylindrical polar coordinates, for the vector field $\underline{F} = \rho^{-2}\hat{\underline{\rho}}$ over the cylinder $0 \leq \rho \leq r_0$, $-1 \leq z \leq 1$ for

   (a) $r_0 = 1$

   (b) $r_0 = 2$

3. If S is the surface of the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, find the surface integral
$$\int\int_S (x\underline{i} + yz\underline{j}) \cdot \underline{dS}$$

   (a) directly

   (b) by using Gauss' theorem

   Hint :- When evaluating directly, show that the unit normal on the sloping face is $\frac{1}{\sqrt{3}}(\underline{i}+\underline{j}+\underline{k})$ and that $\underline{dS} = (\underline{i} + \underline{j} + \underline{k})dxdy$

Answers

1. Both sides are 156,

2. Both sides equal (a) 4π, (b) 2π,

3. (a) $\dfrac{5}{24}$ [only contribution is from the sloping face] (b) $\dfrac{5}{24}$ [by volume integral of $(1+z)$].

3. Green's Identities (3D)

Like Gauss' theorem, Green's identities relate surface integrals to volume integrals. However, Green's identities are concerned with two scalar fields $u(x, y, z)$ and $w(x, y, z)$. Two statements of Green's identities are as follows

$$\iint_S (u\underline{\nabla}w) \cdot \underline{dS} = \iiint_V \left\{\underline{\nabla}u \cdot \underline{\nabla}w + u\underline{\nabla}^2 w\right\} dV \qquad [1]$$

and

$$\iint_S \{u\underline{\nabla}w - v\underline{\nabla}u\} \cdot \underline{dS} = \iiint_V \left\{u\underline{\nabla}^2 w - w\underline{\nabla}^2 u\right\} dV \qquad [2]$$

Proof of Green's identities

Green's identities can be derived from Gauss' theorem and a vector derivative identity.

Vector identity (1) from subsection 6 of 28.2 states that $\underline{\nabla} \cdot (\phi\underline{A}) = (\underline{\nabla}\phi) \cdot \underline{A} + \phi(\underline{\nabla} \cdot \underline{A})$.
Letting $\phi = u$ and $\underline{A} = \underline{\nabla}w$ in this identity,

$$\underline{\nabla} \cdot (u\underline{\nabla}w) = (\underline{\nabla}u) \cdot (\underline{\nabla}w) + u(\underline{\nabla} \cdot (\underline{\nabla}w)) = (\underline{\nabla}u) \cdot (\underline{\nabla}w) + u\underline{\nabla}^2 w$$

Gauss' theorem states

$$\iint_S \underline{F} \cdot \underline{dS} = \iiint_V \underline{\nabla} \cdot \underline{F} dV$$

Now, letting $\underline{F} = u\underline{\nabla}w$,

$$\begin{aligned} \iint_S (u\underline{\nabla}w) \cdot \underline{dS} &= \iiint_V \underline{\nabla} \cdot (u\underline{\nabla}w) dV \\ &= \iiint_V \left\{(\underline{\nabla}u) \cdot (\underline{\nabla}w) + u\underline{\nabla}^2 w\right\} dV \end{aligned}$$

This is Green's identity [1].
Reversing the roles of u and w,

$$\iint_S (w\underline{\nabla}u) \cdot \underline{dS} = \iiint_V \left\{(\underline{\nabla}w) \cdot (\underline{\nabla}u) + w\underline{\nabla}^2 u\right\} dV$$

Subtracting the last two equations yields Green's identity [2].

Key Point 10

Green's Identities

[1] $$\iint_S (u\underline{\nabla}w) \cdot \underline{dS} = \iiint_V \left\{\underline{\nabla}u \cdot \underline{\nabla}w + u\underline{\nabla}^2 w\right\} dV$$

[2] $$\iint_S \{u\underline{\nabla}w - v\underline{\nabla}u\} \cdot \underline{dS} = \iiint_V \left\{u\underline{\nabla}^2 w - w\underline{\nabla}^2 u\right\} dV$$

Example 38

Verify Green's first identity for $u = (x - x^2)y$, $w = xy + z^2$ and the unit cube, $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

As $w = xy + z^2$, $\underline{\nabla} w = y\underline{i} + x\underline{j} + 2z\underline{k}$. Thus $u\underline{\nabla} w = (xy - x^2 y)(y\underline{i} + x\underline{j} + 2z\underline{k})$ and the surface integral is of this quantity (scalar product with $\underline{dS}$) integrated over the surface of the unit cube.

On the three faces $x = 0$, $x = 1$, $y = 0$, the vector $u\underline{\nabla} w = \underline{0}$ and so the contribution to the surface integral is zero.

On the face $y = 1$, $u\underline{\nabla} w = (x - x^2)(\underline{i} + x\underline{j} + 2z\underline{k})$ and $\underline{dS} = dxdz\underline{j}$ so $(u\underline{\nabla} w) \cdot \underline{dS} = (x^2 - x^3)dxdz$ and the contribution to the integral is

$$\int_{x=0}^{1} \int_{z=0}^{1} (x^2 - x^3)dzdx = \int_{0}^{1} (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$

On the face $z = 0$, $u\underline{\nabla} w = (x - x^2)y(y\underline{i} + x\underline{j})$ and $\underline{dS} = -dxdz\underline{k}$ so $(u\underline{\nabla} w) \cdot \underline{dS} = 0$ and the contribution to the integral is zero.

On the face $z = 1$, $u\underline{\nabla} w = (x - x^2)y(y\underline{i} + x\underline{j} + 2\underline{k})$ and $\underline{dS} = dxdy\underline{k}$ so $(u\underline{\nabla} w) \cdot \underline{dS} = 2y(x - x^2)dxdy$ and the contribution to the integral is

$$\int_{x=0}^{1} \int_{y=0}^{1} 2y(x - x^2)dydx = \int_{x=0}^{1} \left[y^2(x - x^2)\right]_{y=0}^{1} dx = \int_{0}^{1} (x - x^2)dx = \frac{1}{6}.$$

Thus, $\displaystyle\iint_S (u\underline{\nabla} w) \cdot \underline{dS} = 0 + 0 + 0 + \frac{1}{12} + 0 + \frac{1}{6} = \frac{1}{4}.$

Now evaluate $\displaystyle\iiint_V \left\{\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w\right\} dV.$

Note that $\underline{\nabla} u = (1 - 2x)y\underline{i} + (x - x^2)\underline{j}$ and $\underline{\nabla}^2 w = 2$ so

$\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w = (1 - 2x)y^2 + (x - x^2)x + 2(x - x^2)y = x^2 - x^3 + 2xy - 2x^2 y + y^2 - 2xy^2$

and the integral

$$\begin{aligned}
\iiint_V \left\{\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w\right\} dV &= \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (x^2 - x^3 + 2xy - 2x^2 y + y^2 - 2xy^2)dxdydz \\
&= \int_{z=0}^{1} \int_{y=0}^{1} \left[\frac{x^3}{3} - \frac{x^4}{4} + x^2 y - \frac{2}{3}x^3 y + xy^2 - x^2 y^2\right]_{x=0}^{1} dydz \\
&= \int_{z=0}^{1} \int_{y=0}^{1} (\frac{1}{12} + \frac{y}{3})dydz = \int_{z=0}^{1} \left[\frac{y}{12} + \frac{y^2}{6}\right]_{y=0}^{1} dz \\
&= \int_{z=0}^{1} (\frac{1}{4})dz = \left[\frac{z}{4}\right]_{z=0}^{1} = \frac{1}{4}
\end{aligned}$$

Hence $\displaystyle\iint_S (u\underline{\nabla} w) \cdot \underline{dS} = \iiint_V \left[\underline{\nabla} u \cdot \underline{\nabla} w + u\underline{\nabla}^2 w\right] dV = \frac{1}{4}$ and Green's first identity is verified.

Green's theorem in the plane

This states that

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

S is a 2-D surface with perimeter C; $P(x, y)$ and $Q(x, y)$ are scalar functions.

This should not be confused with Green's identities.

Justification of Green's theorem in the plane

Green's theorem in the plane can be derived from Stokes' theorem.

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dr}$$

Now let $\underline{F}$ be the vector field $P(x, y)\underline{i} + Q(x, y)\underline{j}$ i.e. there is no dependence on z and there are no components in the $z-$ direction. Now

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underline{k}$$

and $\underline{dS} = dxdy\underline{k}$ giving $(\underline{\nabla} \times \underline{F}) \cdot \underline{dS} = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dxdy$.

Thus Stokes' theorem becomes

$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C \underline{F} \cdot \underline{dr}$$

and Green's theorem in the plane follows.

Key Point 11

Green's Theorem in the Plane

$$\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

This relates a line integral around a closed path C with a double integral over the region S enclosed by C. It is effectively a two-dimensional form of Stokes' theorem.

Example 39

Evaluate the line integral $\oint_C \left[(4x^2 + y - 3)dx + (3x^2 + 4y^2 - 2)dy\right]$ around the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 1$.

Solution

The integral could be obtained by evaluating four line integrals but it is easier to note that $[(4x^2 + y - 3)dx + (3x^2 + 4y^2 - 2)dy]$ is of the form $Pdx + Qdy$ with $P = 4x^2 + y - 3$ and $Q = 3x^2 + 4y^2 - 2$. It is thus of a suitable form for Green's theorem in the plane.

Note that $\dfrac{\partial Q}{\partial x} = 6x$ and $\dfrac{\partial P}{\partial y} = 1$.

Green's theorem in the plane becomes

$$\begin{aligned}\oint_C \{(4x^2 + y - 3)dx + (3x^2 + 4y^2 - 2)dy\} &= \int_{y=0}^{1} \int_{x=0}^{3} (6x - 1)\, dxdy \\ &= \int_{y=0}^{1} \Big[3x^2 - x\Big]_{x=0}^{3} dy = \int_{y=0}^{1} 24\ dy = 24\end{aligned}$$

Example 40

Verify Green's theorem in the plane for the integral $\oint_C \left[4z dy + (y^2 - 2)dz\right]$ and the triangular contour starting at the origin $O = (0, 0, 0)$ and going to $A = (0, 2, 0)$ and $B = (0, 0, 1)$ before returning to the origin.

Solution

The whole of the contour is in the plane $x = 0$ and Green's theorem in the plane becomes

$$\oint_C (Pdy + Qdz) = \iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz$$

(a) Firstly evaluate $\oint_C \left\{4zdy + (y^2 - 2)dz\right\}$.

On OA, $z = 0$ and $dz = 0$. As the integrand is zero, the integral will also be zero.

On AB, $z = (1 - \frac{y}{2})$ and $dz = -\frac{1}{2}dy$. The integral is

$$\int_{y=2}^{0} \left((4 - 2y)dy - \frac{1}{2}(y^2 - 2)dy\right) = \int_2^0 (5 - 2y - \frac{1}{2}y^2)dy = \left[5y - y^2 - \frac{1}{6}y^3\right]_2^0 = -\frac{14}{3}$$

On BO, $y = 0$ and $dy = 0$. The integral is $\displaystyle\int_1^0 (-2)dz = \left[-2z\right]_1^0 = 2.$

Summing, $\displaystyle\oint_C \left(4zdy + (y^2 - 2)dz\right) = -\frac{8}{3}$

(b) Secondly evaluate $\displaystyle\iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz$

In this example, $P = 4z$ and $Q = y^2 - 2$. Thus $\dfrac{\partial P}{\partial z} = 4$ and $\dfrac{\partial Q}{\partial y} = 2y$. Hence,

$$\begin{aligned}
\iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz &= \int_{y=0}^{2} \int_{z=0}^{1-y/2} (2y - 4)\, dzdy \\
&= \int_{y=0}^{2} \left[2yz - 4z\right]_{z=0}^{1-y/2} dy = \int_{y=0}^{2} \left(-y^2 + 4y - 4\right) dy \\
&= \left[-\frac{1}{3}y^3 + 2y^2 - 4y\right]_0^2 = -\frac{8}{3}
\end{aligned}$$

Hence:

$$\oint_C (Pdy + Qdz) = \iint_S \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial z}\right) dydz = -\frac{8}{3}$$ and Green's theorem in the plane is verified.

One very useful, special case of Green's theorem in the plane is when $Q = x$ and $P = -y$. The theorem becomes

$$\oint_C \{-y dx + x dy\} = \iint_S (1 - (-1))\, dx dy$$

The right-hand side becomes $\iint_S 2\, dx dy$ i.e. $2A$ where A is the area inside the contour C. Hence

$$A = \frac{1}{2} \oint_C \{x dy - y dx\}$$

This result is known as the **area theorem**. It gives us the area bounded by a curve C in terms of a line integral around C.

Example 41

Verify the area theorem for the segment of the circle $x^2 + y^2 = 4$ lying above the line $y = 1$.

Solution

Firstly, the area of the segment $ADBC$ can be found by subtracting the area of the triangle $OADB$ from the area of the sector $OACB$. The triangle has area $\frac{1}{2} \times 2\sqrt{3} \times 1 = \sqrt{3}$. The sector has area $\frac{\pi}{3} \times 2^2 = \frac{4}{3}\pi$. Thus segment $ADBC$ has area $\frac{4}{3}\pi - \sqrt{3}$.

Now, evaluate the integral $\oint_C \{x dy - y dx\}$ around the segment.

Along the line, $y = 1$, $dy = 0$ so the integral $\int_C \{x dy - y dx\}$ becomes $\int_{-\sqrt{3}}^{\sqrt{3}} (x \times 0 - 1 \times dx) = \int_{-\sqrt{3}}^{\sqrt{3}} (-dx) = -2\sqrt{3}$.

Along the arc of the circle, $y = \sqrt{4 - x^2} = (4 - x^2)^{1/2}$ so $dy = -x(4 - x^2)^{-1/2} dx$. The integral $\int_C \{x dy - y dx\}$ becomes

$$\begin{aligned}
\int_{\sqrt{3}}^{-\sqrt{3}} \{-x^2(4 - x^2)^{-1/2} - (4 - x^2)^{1/2}\} dx &= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{4}{\sqrt{4 - x^2}} dx \\
&= \int_{-\pi/3}^{\pi/3} 4 \frac{1}{2\cos\theta} 2\cos\theta\, d\theta \text{ (letting } x = 2\sin\theta) \\
&= \int_{-\pi/3}^{\pi/3} 4 d\theta = \frac{8}{3}\pi
\end{aligned}$$

So, $\frac{1}{2} \oint_C \{x dy - y dx\} = \frac{1}{2} \left[\frac{8}{3}\pi - 2\sqrt{3} \right] = \frac{4}{3}\pi - \sqrt{3}$.

Hence both sides of the area theorem equal $\frac{4}{3}\pi - \sqrt{3}$ thus verifying the theorem.

Task

Verify Green's theorem in the plane when applied to the integral

$$\oint_C \{(5x + 2y - 7)dx + (3x - 4y + 5)dy\}$$

where C represents the perimeter of the trapezium with vertices at $(0, 0)$, $(3, 0)$, $(6, 1)$ and $(1, 1)$.

First let $P = 5x + 2y - 7$ and $Q = 3x - 4y + 5$ and find $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$:

Your solution

Answer
1

Now find $\displaystyle\int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$ over the trapezium:

Your solution

Answer
4 (by elementary geometry)

Now find $\displaystyle\int (Pdx + Qdy)$ along the four sides of the trapezium, beginning with the line from $(0, 0)$ to $(3, 0)$, and then proceeding anti-clockwise.

Your solution

Answers $1.5,\ 66,\ -62.5,\ -1$ whose sum is 4.

Finally show that the two sides of the statement of Green's theorem are equal:

Your solution

Answer

Both sides are 4.

Exercises

1. Verify Green's identity [1] (page 73) for the functions $u = xyz$, $w = y^2$ and the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

2. Verify the area theorem for

   (a) The area above $y = 0$, but below $y = 1 - x^2$.

   (b) The segment of the circle $x^2 + y^2 = 1$, to the upper left of the line $y = 1 - x$.

Answers

1. Both integrals in [1] equal $\dfrac{1}{2}$

2. (a) both sides give a value of $\dfrac{4}{3}$, (b) both sides give a value of $\dfrac{\pi}{4} - \dfrac{1}{2}$.